Introduction to homotopy theory

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Abstract. This talk is meant to be a quick introduction to basic homotopy and homology theory, and is designed to be a supplement to Ivan Smith's Part III course on algebraic topology[1]. All of the material here is basically stolen from the three excellent books listed in the bibliography, to which the reader is referred to for more information[2].

[1] Many apologies for the thousands of typos and other assorted errors here. I wrote a lot of these note rather late last night.

[2] I would also like to say now to Part III students who get lost reading this (or indeed, lost at any point during Part III) are more than welcome to email me (email address above); the chances of me being helpful are slim, but hey.

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1 Homotopy

If two spaces are homeomorphic then they are the 'same' as far as algebraic topology is concerned. That is, if X and Y are homeomorphic then any (topological) properties that X has are also properties of Y.

Example 1. The real line $\mathbb{R}$ is homeomorphic to the open interval $(-1,1)$. An explicit homeomorphism is $f:(-1,1)\to\mathbb{R}$, $x\mapsto\frac{x}{1-x^2}$.

Example 2. Stereographic projection exhibits a homeomorphism from $\mathbb{C}$ to $S^2\backslash\{N\}$.

In practice however the (equivalence) relation of homeomorphism is too strong to be of much use. Two spaces can be very 'similar' without being homeomophic, and we thus seek a weaker relation. The correct property to study turns out to be the following:

Definition 1. *Let $f,g:X\to Y$ be maps. Suppose we have a collection of maps $\{f_t:X\to Y : t\in[0,1]\}$ such that $f_0=f$, $f_1=g$ and if $F:X\times I\to Y$ is the map $F(x,t):=f_t(x)$ then F is continuous. We then say f and g are* ***homotopic****, with f_t a* ***homotopy*** *between them. We write $f_0\sim f_1$ to indicate that f and g are homotopic.*

Let us check that the relation $\sim$ is an equivalence relation on the set $C(X,Y)$ of maps $f:X\to Y$. First, clearly $f\sim f$ via the constant homotopy $f_t\equiv f$. If $f\sim g$ via f_t then if $\bar{f}_t:=f_{1-t}$ then $\bar{f}_t$ is a homotopy from g to f. Finally suppose $f\sim g$ and $g\sim h$ via homotopies f_t and g_t respectively. It's obvious what the desired homotopy h_t from f to h must be; simply follow f_t first and then go along g_t. Explicity we can write h_t as

$$h_t:=\begin{cases} f_{2t} & t\in[0,1/2] \\ g_{2t-1} & t\in[1/2,1].\end{cases}$$

To check h_t is continuous (and hence a well defined homotopy) one invokes the following trivial lemma, the proof of which is left as an exercise.

Lemma 1. (the gluing lemma)
Let $X = A \cup B$ with A, B closed. Suppose we have (continuous) maps $f : A \to Y$ and $g : B \to Y$ such that $f \equiv g$ on $A \cap B$. Then the map $h : X \to Y$ defined by

$$h = \begin{cases} f & \text{on } A \\ g & \text{on } B \end{cases}$$

is continuous.

Definition 2. *Given two topological spaces X and Y, we say that X and Y are* ***homotopy equivalent*** *if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \sim \mathbb{1}_X$ and $fg \sim \mathbb{1}_Y$. In this case we say the maps f and g are* ***homotopy equivalences****, and that g is the* ***homotopy inverse*** *to f.*

The relation of being homotopy equivalent is best pictured as follows: two spaces X and Y are homotopy equivalent if X can be continuously deformed (i.e. without tearing) into Y. Thus the solid torus (doughnut) is homotopy eqiuvalent to a coffee cup[3]. However the sphere is not homotopy equivalent to the torus.

It can sometimes be fun[4] to try to construct explicit homotopies. Let's look at some simple examples.

Example 3. $\mathbb{R}^n$ is homotopy equivalent to $\{0\}$. Namely, let $r : \mathbb{R}^n \to \{0\}$ be the map $x \mapsto 0$ and $i : \{0\} \hookrightarrow \mathbb{R}^n$ be the inclusion map. Then $ri = \mathbb{1}_{\{0\}}$ (and so certainly $ri \sim \mathbb{1}_{\{0\}}$) and $ir \sim \mathbb{1}_{\mathbb{R}^n}$ via the homotopy

$$f_t(x) = tx.$$

We say that a space X is **contractible** if it homotopy equivalent to a point. Thus we have just shown $\mathbb{R}^n$ is contractible. Exactly the same argument replacing $\mathbb{R}^n$ with the unit disc $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ shows that D^n is also contractible.

Example 4. The unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^n \backslash \{0\}$. For this take $r : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ by the map

$$r : x \mapsto \frac{x}{|x|}$$

and let $i : S^{n-1} \hookrightarrow \mathbb{R}^n$ be inclusion. Then as before $ri = \mathbb{1}_{S^{n-1}}$, and $ir \sim \mathbb{1}_{\mathbb{R}^n}$ via the homotopy f_t defined by

$$f_t(x) = tx + (1-t)\frac{x}{|x|}.$$

[3] This is the most unoriginal example ever. Check out the wikipedia article on homotopy.
[4] For some definition of 'fun'...

In fact, these are both examples of retractions. Let $A \subseteq X$. A **retraction** $r : X \to X$ is a map such that $r(X) = A$ and $r|_A = \mathbb{1}_A$. We then say A is a **retract** of X. We say that r is a **deformation retraction** if there exists a homotopy r_t from r to $\mathbb{1}_X$.

More generally, a homotopy $f_t : X \to X$ whose restriction to a subspace $A \subseteq X$ is independent of t, i.e. $f_t|_A \equiv f$ for some map $f : A \to X$ is called a homotopy **relative** to A, or for short, a homotopy **rel** A. Thus a deformation retraction of X onto A is a homotopy rel A from the identity map of X to a retraction of X onto A.

2 The fundamental group

We will now define the fundamental group of a topolgical space X. In what follows, we let I denote the unit interval $[0,1]$, and ∂I denote its boundary $\{0,1\}$. A **path** in a space X is simply a map $f : I \to X$. A **loop** is a path such that $f(0) = f(1)$. We will denote the equivalence class of a path f under the equivalence relation $\sim$ by $[f]$.

Definition 3. *Given two paths $f, g : I \to X$ such that $f(1) = g(0)$ we define the **product path** $f \cdot g$ to be the path that goes round f first and then goes round g; explicitly*

$$f \cdot g(s) := \begin{cases} f(2s) & s \in [0,1/2] \\ g(2s-1) & s \in [1/2,1]. \end{cases}$$

Lemma 2. *The operation '$\cdot$' is well defined on the level of homotopy classes, that is,*

$$[f] \cdot [g] = [f \cdot g]$$

is well defined.

Proof. We need to show that if $f_0 \sim f_1$ via f_t and $g_0 \sim g_1$ via g_t then $f_0 \cdot g_0 \sim f_1 \cdot g_1$. But this is clear; the desired homotopy is simply $f_t \cdot g_t$.

Here is the general plan: fix a point $x \in X$ and let $\pi_1(X,x)$ denote the set of homotopy classes of loops based at x:

$$\pi_1(X,x) = \{[f] \,:\, f : I \to X, f(0) = f(1) = x\}.$$

We want to make $\pi_1(X,x)$ into a group, called the **fundamental group of X at x** with multiplication $[f] \cdot [g] = [f \cdot g]$. The key tool we use in order to do this is the following useful lemma.

Lemma 3. (the reprametrisation lemma)

Let $\varphi_1, \varphi_2 : I \to I$ be maps such that $\varphi_i(\partial I) \subseteq \partial I$ for $i = 1,2$ such that $\varphi_1|_{\partial I} = \varphi_2|_{\partial I}$. Then for any path $f : I \to X$, we have

$$f\varphi_1 \sim f\varphi_2 \quad \textit{rel } \partial I.$$

Proof. The desired homotopy is

$$f_t(x) := tf\varphi_2(x) + (1-t)f\varphi_1(x).$$

This is well defined as $I \times I$ is convex, and it is easy to check f_t has the desired properties.

In practice we will normally apply the reparametrisation lemma with $\varphi_2 = \mathbb{1}_I$; that is, to show $f \sim g$ we find a map φ_1 such that $g = f\varphi_1$, and then take $\varphi_2 = \mathbb{1}_I$.

Definition 4. *Given paths $f_1, \ldots, f_n : I \to X$ such that $f_i(1) = f_{i+1}(0)$ we define the product path $f_1 \cdots f_n$ to be the path that goes round f_i in the time interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$. Explicitly*

$$f_1 \cdots f_n(s) := f_i(ns - i + 1) \text{ for } s \in \left[\frac{i-1}{n}, \frac{i}{n}\right].$$

Lemma 4. (associativity)
Suppose $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are paths such that $f_1(0) = g_1(0)$ and that the product paths $f_1 \cdots f_n$ and $g_1 \cdots g_n$ are defined, and finally that $f_i \sim g_i$ rel ∂I via the homotopy f_t^i. Then

$$f_1 \cdots f_n \sim g_1 \cdots g_n \quad \text{rel } \partial I.$$

Moreover for any $1 \leq i \leq n-1$ we have

$$(f_1 \cdots f_i) \cdot (f_{i+1} \cdots f_n) \sim f_1 \cdots f_n \quad \text{rel } \partial I.$$

Proof. For the first statement, the required homotopy is unsuprisingly $f_t^1 \cdots f_t^n$. For the second statement, let φ_i be the map taking the points $0, 1/2, 1$ onto $0, i/n, 1$ respectively, and that is linear in between. Then

$$(f_1 \cdots f_i) \cdot (f_{i+1} \cdots f_n) = (f_1 \cdots f_n)\varphi_i;$$

now apply the reparametrisation lemma.

Define the constant path $e_x : I \to X$ by $e_x(s) = x$ for all $s \in I$.

Lemma 5. (identity)
Let f be a loop based at x. Then $e_x \cdot f \sim f \sim f \cdot e_x$. Hence $[e_x] \cdot [f] = [f] = [f] \cdot [e_x]$, and so $[e_x]$ is the identity element of $\pi_1(X, x)$ under '$\cdot$'.

Proof. To see $e_x \cdot f \sim f$, consider the map φ_1 that takes $0, 1/2, 1$ onto $0, 0, 1$ respectively.Then

$$e_x \cdot f = f\varphi_1.$$

The reparametrisation lemma then shows $e_x \cdot f \sim f$. To see $f \cdot e_x \sim f$, consider φ_2 the map that takes $0, 1/2, 1$ onto $0, 1, 1$ respectively.

It thus remains to see the existence of inverses. Given any path $f : I \to X$, let $\bar{f}$ denote the path

$$\bar{f}(s) := f(1-s).$$

Lemma 6. (inverses)
For any loop $f : I \to X$ we have $f \cdot \bar{f} \sim e_x \sim \bar{f} \cdot f$. Hence $[f] \cdot [\bar{f}] = [e_x] = [\bar{f}] \cdot [f]$ and $[\bar{f}]$ is the inverse of $[f]$ under '$\cdot$'.

Proof. To see $f \cdot \bar{f} \sim e_x$ let φ_1 be the map taking $0, 1/2, 1$ onto $0, 1, 0$, and let $\varphi_2 \equiv 0$. Then

$$f \cdot \bar{f} = f\varphi_1 \quad \text{and} \quad f\varphi_2 = e_x.$$

For $\bar{f} \cdot f \sim e_x$, let φ_3 take $0, 1/2, 1$ onto $1, 0, 1$ and $\varphi_4 \equiv 1$. Then

$$\bar{f} \cdot f = f\varphi_3 \quad \text{and} \quad f\varphi_4 = e_x.$$

This finally completes the proof of:

Theorem 1. (the fundamental group)
Let X be a topological space and $x \in X$. Let $\pi_1(X, x)$ denote the set of homotopy classes of loops based at x. Then $\pi_1(X, x)$ admits a group structure under the multiplication '$\cdot$'.

We now wish to investigate the dependence of $\pi_1(X, x)$ on the basepoint x. In fact, the following holds:

Lemma 7. *Let x_0 and x_1 be points in the same path component of X. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Let h be a path from x_0 to x_1, and for a loop f based at x_0, define

$$\beta_h(f) := \bar{h} \cdot f \cdot h,$$

so $\beta_h(f)$ is a loop based at x_1. Note that if $f_0 \sim f_1$ via f_t then $\beta_j(f_0) \sim \beta_h(f_1)$ vai $\bar{h} \cdot f_t \cdot h$, and thus $\beta(h)$ induces a map $\pi_1(X, x_0) \to \pi_1(X, x_1)$. This map is a group homomorphism, since

$$\begin{aligned} \beta_h([f]) \cdot \beta_h([g]) &= [\bar{h} \cdot f \cdot h] \cdot [\bar{h} \cdot g \cdot h] \\ &= [\bar{h} \cdot f \cdot g \cdot h] \\ &= \beta_h([f \cdot g]). \end{aligned}$$

Morover β_h is an isomorphism, since $\beta_{\bar{h}} : \pi_1(X, x_1) \to \pi_1(X, x_0)$, $[f] \mapsto [h \cdot f \cdot \bar{h}]$ is visibly an inverse.

Remark 1. The isomorphism we have just constructed is not 'natural'; it depends on the path class $[h]$ of h. Note that in the special case $x_0 = x_1$, β_h is an inner automorphism of $\pi_1(X, x_0)$.

As a consequence of the preceeding lemma we will often write $\pi_1(X)$ instead of $\pi_1(X, x)$ when X is path connected, since the choice of basepoint is somewhat unimportant.

Suppose now $\psi : X \to Y$, with $\psi(x) = y$. We wish to define an induced homomorphism $\psi_* : \pi_1(X, x) \to \pi_1(Y, y)$.

Lemma 8. *Define* $\psi_* : \pi_1(X, x) \to \pi_1(Y, y)$ *by*

$$\psi_*([f]) = [\psi f] \quad \textit{for } f : I \to X \textit{ a loop based at } x.$$

Then ψ_* *is a well defined group homomorphism satisfying the following* ***functorial properties****:*

1. $\mathbb{1}_* = \mathbb{1}$,
2. *if* $\psi : X \to Y$ *and* $\varphi : Y \to Z$ *then* $(\varphi\psi)_* = \varphi_*\psi_*$.

Proof. ψ_* is well defined as if $f_0 \sim f_1$ via f_t then $\psi f_0 \sim \psi f_1$ via ψf_t. ψ_* is a homomorphism as $\psi(f \cdot g) = \psi f \cdot \psi g$, both taking the value $\psi f(2s)$ for $s \in [0, 1/2]$ and $\psi g(2s - 1)$ for $s \in [1/2, 1]$. It is immediate that $\mathbb{1}_* = \mathbb{1}$, and $(\varphi\psi)_* = \varphi_*\psi_*$ follows from the fact that composition of maps is associative.

Corollary 1. *Suppose* $\psi : X \to Y$ *is a homeomorphism. Then* $\pi_1(X, x) \cong \pi_1(Y, y)$.

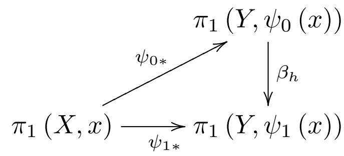
Proof. ψ_* is an isomorphism, with inverse $\left(\psi^{-1}\right)_*$.

We now wish to strengthen the preceeding corollary to the statement that if ψ is a homotopy equivalence then $\pi_1(X, x) \cong \pi_1(Y, y)$. In order to do so we will need the following technical lemma.

Lemma 9. *Let* $\psi_t : X \to Y$ *be a homotopy between two maps* ψ_0 *and* ψ_1. *Fix a point* $x \in X$, *and let* h *be the path in* Y *defined by*

$$h(t) = \psi_t(x).$$

Then the following diagram commutes.



Proof. Define $h_s(t) = h((1 - s)t + s)$. Then if f is a loop based at x, $\bar{h}_t \cdot \psi_t f \cdot h_t$ is a homotopy of loops based at $\psi_1(x)$. For $t = 0$, this is $\bar{h} \cdot \psi_0 f \cdot h = \beta_h(\psi f)$, and for $t = 1$ this is $\psi_1 f$.

Theorem 2. *If* $\psi : X \to Y$ *is a homotopy equivalence then* $\pi_1(X, x) \cong \pi_1(Y, \psi(x))$.

Proof. Let $\varphi : Y \to X$ be a homotopy inverse, so $\varphi\psi \sim \mathbb{1}_X$ and $\psi\varphi \sim \mathbb{1}_Y$. Consider the composition

$$\pi_1(X, x) \xrightarrow{\psi_*} \pi_1(Y, \psi(x)) \xrightarrow{\varphi_*} \pi_1(X, \varphi\psi(x)) \xrightarrow{\psi_*} \pi_1(Y, \psi\varphi\psi(x)) .$$

The composition of the first two maps is an isomorphism, as $\varphi_*\psi_* = \beta_h \cdot \mathbb{1}_* = \beta_h$ by the previous lemma, which is an isomorphism. Thus ψ_* is injective and φ_* is surjective. Similarly, by considering the composition of the last two maps we see φ_* is injective and ψ_* is surjective, which thus completes the proof.

Definition 5. *We say that a path connected space X is* ***simply connected*** *if* $\pi_1(X) = 0$.

Corollary 2. $\mathbb{R}^n$ *is simply connected.*

Proof. It is not difficult to see that $\pi_1(\{0\}) = 0$. Since we have shown that $\mathbb{R}^n$ is homotopy equivalent to $\{0\}$, the result follows.

3 Computations of the fundamental group

In this section we will first compute $\pi_1(S^1)$. We will then show that $\pi_1(S^n) = 0$ for $n > 1$.We will then state (without proof) an extremely powerful theorem that allows us to compute the fundamental group of a space $X = U \cup V$ in terms of the fundamental groups of U, V and $U \cap V$. We will use this to compute the fundamental group of the Klein bottle.

Theorem 3. *The fundamental group of the circle is the integers:* $\pi_1(S^1) = \mathbb{Z}$.

At a crucial stage in the proof below we shall appeal to a result from covering spaces, which will be proved in the next section.

Proof. Define a map $p : \mathbb{R} \to S^1$ by $p(t) = e^{2\pi i t}$. Let f be any loop in S^1 based at 1. Then by the monodromy theorem (see the next section) there exists a unique map $\tilde{f} : I \to \mathbb{R}$ with $\tilde{f}(0) = 0$ such that $p\tilde{f} = f$ (we call such a map $\tilde{f}$ a **lifting** of f). Hence $\tilde{f}(1) \in p^{-1}(1) = \mathbb{Z}$, and we may define the **winding number** $w(f)$ to be the integer

$$w(f) = \tilde{f}(1) .$$

We claim that w induces an isomorphism $\pi_1(S^1, 1) \to \mathbb{Z}$. The monodromy theorem guarantees that w is well defined on the level of homotopy classes, that is, if $f \sim g$ then $\tilde{f}(1) = \tilde{g}(1)$. To check that w is a homomorphism, suppose f and g are two loops with liftings $\tilde{f}$ and $\tilde{g}$ with $\tilde{f}(0) = \tilde{g}(0) = 0$, and $w(f) = \tilde{f}(1) =: n$ and $w(g) = \tilde{g}(1) =: m$. Define $\tilde{h}(t) := \tilde{g}(t) + n$. Then $\tilde{h}(0) = n = \tilde{f}(1)$, and so $\tilde{f} \cdot \tilde{h}$ is defined and is a lifting of $f \cdot g$, with $\tilde{f} \cdot \tilde{h}(0) = 0$. Hence

$$w([f \cdot g]) = \tilde{f} \cdot \tilde{h}(1) = \tilde{h}(1) = \tilde{g}(1) + n = m + n = w([f]) + w([g]) .$$

w is surjective since if $n \in \mathbb{Z}$ and $\tilde{f} : I \to \mathbb{R}$ is the path $\tilde{f}(t) = tn$ then $f := p\tilde{f}$ is a loop in S^1 which has $w([f]) = n$. Finally w is injective, since if $w([f]) = 0$ then the lifting $\tilde{f}$ of f with $\tilde{f}(0) = 0$ is a loop in $\mathbb{R}$. But as $\pi_1(\mathbb{R}) = 0$ we thus have $[f] = p_*(0) = 0$, since p_* is a group homomorphism.

The situation is very different for $n > 1$ though.

Theorem 4. $\pi_1(S^n) = 0$ *for* $n > 1$.

Proof. Fix a point $x \in S^n$ and let f be a loop based at x. Suppose we could show that there exists $y \in S^n$ such that $f(S^n) \subseteq S^n \backslash \{y\}$. Then since $S^n \backslash \{y\}$ is homeomorphic to $\mathbb{R}^n$ and $\pi_1(\mathbb{R}^n) = 0$, it follows that f is **nullhomotopic**, that is, f is homotopic to a constant map.

So here is the plan. Select any point $y \neq x \in S^n$, and choose a small disc D containing y but not x. We will homotope f to a map f' such that $f \equiv f'$ on $S^n \backslash D$, and $f'(S^n) \subseteq S^n \backslash \{y\}$. This will complete the proof.

For this, note that $f^{-1}D$ is an open subset of $(0, 1)$ and hence is a (possibly infinite) collection of disjoint intervals. The compact set $f^{-1}(y)$ is thus contained in finitely many of them. For each one of these intervals (a_i, b_i), we simply push f away from x but staying within D to acheive the desired homotopy.

The fundamental group behaves well with respect to products; that is, $\pi_1(X \times Y)$ is related to $\pi_1(X)$ and $\pi_1(Y)$ in the most obvious way possible. See Exercise 2 below.

We now want to obtain a result that allows us to compute the fundamental group of a space X built out of smaller spaces U, V but not necessarily just a product $U \times V$. This is a very common technique in algebraic topology, and one that we will see again later when we discuss homology.

The next result, a very special case of **Van Kampen's theorem** is one we will only state; for a proof the reader is referred to , p43. The statement uses the concept of the **free product** $G * H$ of two groups G and H. It is defined to be the set of 'reduced words'

$$w = x_1 x_2 \ldots x_n$$

where each x_i lies in G or H, no $x_i = 1$ and adjacent x_i's are not both in G or H. The words are multiplied by juxtaposition and then reduction: after juxtaposition the last x_i in the first word may be in the same group as the first x_i in the second word, and then they must be combined; this combination may cancels those x's out, etc. Thus unit element is defined to be the empty word. The proof that this does indeed define a group is somewhat messy, but essentially obvious and so we omit it.

Theorem 5. (a very special case of Van Kampen's theorem)

Let $X = U \cup V$, *where* U, V *and* $U \cap V$ *are path connected and* $U \cap V \neq \emptyset$. *Pick a basepoint* $x \in U \cap V$.

1. *Suppose* $U \cap V$ *is simply connected. Then*

$$\pi_1(X, x) \cong \pi_1(U, x) * \pi_1(V, x).$$

2. Suppose instead that V simply connected. Then $\pi_1(X,x) \cong \pi_1(U,x)/N$, where N is the normal subgroup of $\pi_1(U,x)$ generated by the image of $\pi_1(U \cap V, x)$.

Example 5. This allows us to compute the fundamental group of the 'figure eight' space $\mathbf{8}$ to be $\mathbb{Z}*\mathbb{Z}$: $\pi_1(\mathbf{8}) = \mathbb{Z}*\mathbb{Z}$. Indeed, the space $\mathbf{8}$ is the union of two circles with whiskers (making them open sets) and with contractible intersection.

We can also use this theorem to compute the fundamental group of a more complicated example.

Example 6. **(the fundamental group of the Klein bottle K)**
Let K denote the Klein bottle, which can be formed by making the following identifications on a square[5].

Let V denote a small disc in the centre, and let U denote $K\backslash\{x\}$, where $x \in V$. Then V is contractible, $U \cap V$ is a circle, and U contracts onto $\mathbf{8}$. The circle $U \cap V$ deforms onto $\mathbf{8}$ and represents the word $aba^{-1}b$ there. It thus follows from the previous corollary that if then

$$\begin{aligned}\pi_1(K,x) &\cong \pi_1(\mathbf{8},x)/N \\ &= \left\langle a, b \,:\, aba^{-1}b = 1 \right\rangle.\end{aligned}$$

These last examples highlight an extremely important point, which is of course clear :

The fundamental group is not necessarily abelian.

4 Covering spaces

In this section we will give an extremely brief introduction to covering spaces, going just far enough to explain the statement of the monodromy theorem, the result that was quoted in the proof of $\pi_1(S^1) = \mathbb{Z}$ in the previous section.

Definition 6. *A **covering space** of a space[6] X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the following condition: there exists an open cover $\{U_\alpha\}_\alpha$ of X such that for each α, $p^{-1}U_\alpha$ is a disjoint union of of nonempty open sets $\{V_{\alpha,i}\}_i$ such that $p|_{V_{\alpha,i}} : V_{\alpha,i} \to U_\alpha$ is a homeomorphism for each i.*

Example 7. The most obvious example of a covering space is the following: let X be any space and set $\tilde{X} = \bigsqcup_{\alpha=1}^{n} X_\alpha$, where each X_α is a copy of X, and $p : \tilde{X} \to X$ is the obvious map.

[5] The picture referred to here will be drawn up on the board during the talk. It can also be found in , p162.

[6] Actually one generally requires that both X and $\tilde{X}$ are Hausdorff, path connected and locally path connected. For the purposes of this talk however, one would do best to ignore these technical nuisances.

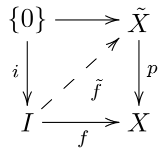
We say a covering space $p : \tilde{X} \to X$ is an n**-sheeted covering** if $\left|p^{-1}(x)\right| = n$ for all $x \in X$. Thus the covering space from the example above has rank n. The rank is not necessarily finite though:

Example 8. Consider the map $p : \mathbb{R} \to S^1$ defined by sending $t \mapsto e^{i\pi t}$. To check this is a covering space, simply take as our open cover $\{U_\alpha\}$ to consist of any two open arcs whose union is S^1. In this case $p^{-1}(x) \cong \mathbb{Z}$ for each point $x \in S^1$, and so p has infinite rank.

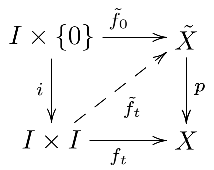
The first main theorems in the theory of covering spaces is the following **lifting theorems**. Proofs of these theorems can be found, for example, in (Br), Theorems III.3.3, III.3.4, or (Ha), Proposition 1.30.

Theorem 6. (the lifting theorems)
Let $p : \tilde{X} \to X$ be a covering space.

1. *If $f : I \to X$ is a path in X and $\tilde{x} \in p^{-1}(f(0))$ then there exists a unique **lift** $\tilde{f}$ of f with $\tilde{f}(0) = \tilde{x}$, that is, a map $\tilde{f} : I \to \tilde{X}$ such that $f = p\tilde{f}$. In other words, we can extend the following diagram uniquely. where i is the inclusion:*



2. *If f_0, f_1 are paths in X, $\tilde{f}_0$ is a lift of f_0 and f_t is a homotopy from f_0 to f_1, then there exists a unique homotopy $\tilde{f}_t$ in $\tilde{X}$ lifting f_t from $\tilde{f}_0$ to a path $\tilde{f}_1$ in $\tilde{X}$ (which is thus automatically a lift of f_1). In other words, there exists a unique $\tilde{f}_t$ such that the following commutes:*



Using these results we can state and prove the theorem quoted in the last section.

Theorem 7. (the monodromy theorem)
Let $p : \tilde{X} \to X$ be a covering space, $f_0, f_1 : I \to X$ two paths in X such that $f_0 \sim f_1$ rel ∂I. Suppose we have two lifts $\tilde{f}_0, \tilde{f}_1$ of f_0, f_1 respectively such that $\tilde{f}_0(0) = \tilde{f}_1(0)$. Then $\tilde{f}_0 \sim \tilde{f}_1$ rel ∂I; in particular $\tilde{f}_0(1) = \tilde{f}_1(1)$.

Proof. Suppose f_t is a homotopy from f_0 to f_1. By the second lifting theorem above, we may lift f_t to a homotopy $\tilde{f}_t$ from $\tilde{f}_0$ to some other path $\tilde{g}$. To complete the proof we need to show that actually $\tilde{g} = \tilde{f}_1$. Since $f_0 \sim f_1$ rel ∂I, the path $t \mapsto f_t(0)$ is a constant path in X. The constant path $t \mapsto \tilde{f}_0(0)$ is a path lifting $t \mapsto f_t(0)$; by the uniqueness part of the first lifting theorem, this must be equal to the path $t \mapsto \tilde{f}_t(0)$. In other words, $t \mapsto \tilde{f}_t(0)$ is a constant path. Similarly $t \mapsto \tilde{f}_t(1)$ is a constant path. This proves the theorem.

The preceeding proof hinged upon the fact that:

the only lift of a constant path is a constant path.

Another simple application of this is the following:

Lemma 10. *Let $p : \tilde{X} \to X$ be a covering space and $\tilde{x} \in \tilde{X}$, with $p(\tilde{x}) =: x$. Then the induced map $p_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective.*

Proof. If $[\tilde{f}_0] \in \ker p$ then there is a homotopy f_t from $f_0 := p\tilde{f}_0$ to the trivial loop e_x. By the second lifting theorem, f_t lifts to a homotopy $\tilde{f}_t$ from $\tilde{f}_0$ to a lift of e_x. Since the only lift of e_x is a constant loop, it follows that $\tilde{f}_1 = e_{\tilde{x}}$, and hence $[\tilde{f}_0] = [e_{\tilde{x}}] = 0 \in \pi_1(\tilde{X}, \tilde{x})$.

We now state without proof the following crucial theorem. Its proof can be found in

Theorem 8. (the universal covering space)

Let X be a topological space[7]*. Then there exists a unique covering space $p : \tilde{X} \to X$ with $\tilde{X}$ simply connected. We call $\tilde{X}$ the* ***universal covering space*** *of X. Moreover if p is an n-sheeted covering then $\pi_1(X)$ has order n.*

Using this theorem we can compute more fundamental groups. Here is an example.

Example 9. $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ for $n \geq 2$, and $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$.

We may realise $\mathbb{R}P^n$ as $S^n / \sim$ where $x \sim -x$ for $x \in S^n$. For $n = 1$, this simply gives $\mathbb{R}P^1 \cong S^1$, so $\pi_1(\mathbb{R}P^1) \cong \pi_1(S^1) = \mathbb{Z}$. For $n \geq 2$ we consider the natural quotient map $p : S^n \to \mathbb{R}P^n$.

We want to show that $p : S^n \to \mathbb{R}P^n$ is a covering map. For this consider the antipodal map $a : S^n \to S^n$ sending $x \mapsto -x$. Note that a is a homeomorphism, and thus $U \subseteq S^n$ is open if and only if $a(U)$ is open. Since $p^{-1}(p(U)) = U \cup a(U)$, we see that p is an open map. Now given a point $x \in \mathbb{R}P^n$, choose a point $\tilde{x} \in S^n$ such that $p(\tilde{x}) = x$. Let U denote the intersection of S^n with the open ball of radius $1/2$ about $\tilde{x}$. Then U contains no antipodal points, and so $U \cap a(U)$ is empty and thus $p : U \to p(U)$ and $p : a(U) \to p(U)$ are bijective, continuous and open; thus homeomorphisms. Thus p is a covering map, as claimed.

Since $\pi_1(S^n) = 0$ for $n \geq 2$, S^n is the universal covering space of $\mathbb{R}P^n$. Since $|p^{-1}(x)| = 2$ for all $x \in \mathbb{R}P^n$, we have $|\pi_1(\mathbb{R}P^n)| = 2$, and thus $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$.

[7] Again, one actually requires in addition that X is Hausdorff, path connected, locally path connected and semi-locally simply connected. Again, my advice to the reader is to ignore these annoying technicalities.

5 The higher homotopy groups

Let us just quickly show how the definition of $\pi_1(X)$ generalises to give us the higher homotopy groups $\pi_n(X)$ for $n > 1$. The idea is essentially the same. Instead of considering loops $f : I \to X$, we consider maps $f : I^n \to X$, where $I^n = [0,1]\overbrace{\times \cdots \times}^{n \text{ times}}[0,1]$ all mapping the boundary ∂I^n to some fixed point $x \in X$.

Given such a map f, we let $[f]$ denote its homotopy class, where homotopies $f_t : I^n \to X$ are required to satisfy $f_t(\partial I^n) = x$ for all t.

As an analogue of our previous product operation '·' we now define for two maps $f, g : I^n \to X$ mapping ∂I^n to x the map $f \cdot g : I^n \to X$ defined by

$$f \cdot g(s_1, \ldots, s_n) := \begin{cases} f(2s_1, s_2 \ldots, s_n) & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \ldots, s_n) & s_1 \in [1/2, 1]. \end{cases}$$

Since only the first coodinate is used in the definition of '·', all the previous arguments used to show that π_1 is a group go through to show that '·' is well defined on the level of homotopy classes, and makes $\pi_n(X, x)$ into a group. The identity element is the equivalence class of the map $e_x : I^n \to X$ sending everything to x, and the inverse of $[f]$ is $\left[\bar{f}\right]$, where $\bar{f}$ is defined by

$$\bar{f}(s_1, \ldots, s_n) := f(1 - s_1, s_2, \ldots, s_n).$$

The first difference between π_1 and π_n for $n > 1$ is the following:

Theorem 9. *For $n > 1$, $\pi_n(X, x)$ is always abelian.*

Proof. Let $f, g : I^n \to X$ be two maps carrying ∂I^n onto $x \in X$. We show $f \cdot g \sim g \cdot f$. This is most easily seen by the following diagram[8].

In words, we can write out the homotopy as follows. First, shrink the domains of f and g to smaller subcubes of I^n, making sure that the region of I^n outside these two smaller subcubes everything is mapped to x. Now since $n > 1$, we can slide the subcubes about in I^n, keeping them disjoint, such that eventually they have switched positions. To conclude, grow the (now flipped round) subcubes out so that once again they fill the square.

A similar argument to the case $n = 1$ proves that if x_0 and x_1 lie in the same path component of x then we have $\pi_n(X, x_0) \cong \pi_n(X, x_1)$. Thus as before we will sometimes simply write $\pi_n(X)$ for path connected spaces X, since the choice of basepoint is unimportant. As before, using exactly the same definition we can define the induced map $\psi_* : \pi_n(X) \to \pi_n(Y)$ for a map $\psi : X \to Y$, and the same arguments show that ψ_* is a groups homomorphism and the association is functorial, that is, $(\varphi\psi)_* = \varphi_*\psi_*$ and $\mathbb{1}_* = \mathbb{1}$. Finally, the same proof goes through to show that if $\psi : X \to Y$ is a homotopy equivalence then

[8] This diagram will be drawn on the board during the talk - it can also be found on p340 of .

$\psi_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism.

A second main difference between π_n for $n > 1$ and π_1 is that in general there is no analogue of Van Kampen's theorem (or indeed, the special case stated above). Thus there is no easy way to compute $\pi_n(X)$ for a space $X = U \cup V$ from knowledge of $\pi_n(U), \pi_n(V)$ etc. Thus in general the groups $\pi_n(X)$ are very hard to compute for $n > 1$.

6 Homology

We will now give a very brief introduction to **homology**, the 'other' side to algebraic topology. The definition of the homology groups $H_n(X)$ are rather less intuitive then the definition of the homotopy groups $\pi_n(X)$, and uses considerable more machinery. The payoff however is that homology groups are in general much easier to compute.

Via the map

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0),$$

each space $\mathbb{R}^n$ is included in $\mathbb{R}^{n+1}$, and thus viewing each $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ we may consider the union

$$\mathbb{R}^\infty = \bigcup_{n \geq 0} \mathbb{R}^n.$$

If P_i denotes the ith standard basis vector in $\mathbb{R}^\infty$, that is, the vector with ith component equal to 1 and the rest all zero, and P_0 denotes the origin then we define the **standard n-simplex** Δ_n to be

$$\Delta_n := \left\{ \sum_{j=0}^{n} t_j P_j \,:\, \sum_{j=0}^{n} t_j = 0, t_j \geq 0 \right\}.$$

Then if X is a topological space, we define a **singular n-simplex** in X to be a continuous map $\sigma : \Delta_n \to X$, and a **singular n-chain** c to be a finite linear combination of singular n-simplices. These collectively form an abelian group $C_n(X)$. We wish to define a map $d : C_n(X) \to C_{n-1}(X)$ given by 'taking alternating sums of the edges of simplices'. More precisely, suppose σ is a simplex. The **faces** of σ are the restrictions of σ to one of the faces of the simplex Δ_n, something which is best pictured pictorially[9]. Thus we may regard each face as an $(n-1)$-simplex in X, and thus the alternating sum of the faces is an element of $C_{n-1}(X)$ as required. Again this is easiest seen pictorially. Below however, we give a way of writing this down formally.

[9] Some helpful pictures will be drawn on the board at this point.

For $0 \le i \le n-1$ we define the n**th face map** $\partial_n^i : \Delta_{n-1} \to \Delta_n$ to be the map

$$\sum_{j=0}^{n-1} t_j P_j \mapsto \sum_{j=0}^{i-1} t_j P_j + \sum_{j=i+1}^{n} t_{j-1} P_j.$$

This induces a map $d : C_n(X) \to C_{n-1}(X)$ by

$$d\sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \partial_n^i.$$

It is not hard to see that $d^2 = 0$.

We call a chain c a **cycle** if $dc = 0$ and a **boundary** if $db = c$ for some chain b then

every boundary is a cycle.

As a consequence it makes sense to speak of 'cycles modulo boundaries'. This is precisely the definition of the homology groups of X. More formally:

Definition 7. *Let X be a topological space and $C_*(X) = \bigoplus_{n \ge 0} C_n(X)$ the set of singular chains in X. Let $Z_n(X) \subseteq C_n(X)$ denote the set of cycles, that is,*

$$Z_n(X) := \{c \in C_n(X) \, : \, dc = 0\},$$

and let $B_n(X) \subseteq Z_n(X)$ denote the set of boundaries, that is,

$$B_n(X) := \{db \in C_n(X) \, : \, b \in C_{n+1}(X)\}.$$

Then since $C_n(X), Z_n(X)$ and $B_n(X)$ are vector spaces (generally infinite dimensional) we can form the quotient vector space

$$H_n(X) := Z_n(X) / B_n(X).$$

*We call $H_n(X)$ the n**th homology group of** X.*

The correct way to think about this is the following: an n-cycle is a boundary unless it encloses an 'n-dimensional hole'. Thus the nth homology group $H_n(X)$ measures the number of n-dimensional holes in X. For example, a sphere S^n evidently has 1 n-dimensional hole and no m-dimensional holes for $m \ne n$. Thus we see $H_n(S^n) = \mathbb{Z}$ and $H_m(S^n) = 0$ for $m \ne n$. Actually this doesn't quite work - as we will see below $H_0(S^n) = \mathbb{Z}$ for all $n \ge 1$; in order to make this correspondence work perfectly one tweaks the definition of homology to obtain the **reduced homology** $\tilde{H}_n(X)$, and with this we can say[10]

$$\tilde{H}_n(X) = \mathbb{Z}^{\#(n\text{-dimensional holes in } X)}.$$

[10] Needless to say there are several occasions in this talk where I am not being entirely rigorous. This is one of them. Rest assured that by the end of Ivan's course you will all know a perfectly rigorous proof of this fact.

It is essentially immediate from the definition of homology that it is a homeomorphism invariant, that is, if X is homeomorphic to Y then $H_n(X) = H_n(Y)$ for all n. What is much less obvious however is that it is a homotopy invariant, that is, if X is homotopy equivalent to Y then $H_n(X) = H_n(Y)$. We will take this on faith for this talk; it will be proved in the algebraic topology course.

The power of homology groups is that once they have been defined the entire machinery of homological algebra can be employed to aid in their computation. An example, which will be discussed in detail in Ivan's course is the **Mayer-Vietoris sequence**, which provides a way of computing the homology groups of $X = U \cup V$ in terms of the homology of U, V and $U \cap V$. Unlike the case of homotopy however, the Mayer-Vietoris works for all $H_n(X)$, not just $H_1(X)$. Here however we will not attempt to cover any of this machinery at all. This has the unfortunate consequence that we have difficulty in computing any homology groups at all! All is not lost however. In the rest of this section we will show how to compute $H_0(X)$ for any space X.

Theorem 10. *Let X be a topological space. Then $H_0(X) = \mathbb{Z}^n$, where n is the number of path components of X.*

Proof. It is immediate that if $X = \coprod_\alpha X_\alpha$ then $H_*(X) = \bigoplus_\alpha H_*(X_\alpha)$, since $\operatorname{im}\{\sigma : \Delta_n \to X\}$ is always contained in a single path component for continuous σ. Thus it is enough to show that for a path connected space X, we have $H_0(X) = \mathbb{Z}$.

So let X be path connected, and define a map $\varepsilon : C_0(X) \to \mathbb{Z}$ by $\varepsilon\left(\sum h_i\sigma_i\right) = \sum h_i$. Clearly ε is surjective. Moreover, $B_0(X) \subseteq \ker\varepsilon$, as if $\tau : [v_0, v_1] \to X$ is a 1-simplex in X with $\tau(v_i) =: x_i \in X$ then if $\sigma_i : \Delta_0 = \{0\} \to X$ has image $x_i \in X$, we have $\varepsilon(d\tau) = \varepsilon(\sigma_1 - \sigma_0) = 1 - 1 = 0$. Thus ε descends to a well defined map $\varepsilon : H_0(X) \to \mathbb{Z}$. We now show $B_0(X) = \ker\varepsilon$, so ε is an isomorphism.

Indeed, if $\varepsilon\left(\sum h_i\sigma_i\right) = 0$, pick a basepoint $x_0 \in X$, write x_i for the image of $\sigma_i : \Delta_0 \to X$. Then define $\tau_i : [v_0, v_i] \to X$ by $\tau_i(v_j) = x_j$ $(j = 0, i)$. Then $d\left(\sum h_i\tau_i\right) = \sum h_i\sigma_i - \left(\sum h_i\right)\sigma_0$, where σ_0 has image x_0 . But $\sum h_i = \varepsilon\left(\sum h_i\sigma_i\right) = 0$ and thus $d\left(\sum h_i\tau_i\right) = \sum h_i\sigma_i$.

7 The Hurewicz isomorphism - relating π_1 to H_1

In this section we will state and prove a fundamental theorem linking $\pi_1(X)$ to $H_1(X)$. For convenience, throughout this section we assume X is **path connected**. Let us fix a basepoint $x \in X$. Any path in X may be regarded as a 1-simplex in X; a loop is thus a cycle. Thus any loops f at x represents a homotopy class $[f] \in \pi_1(X, x)$ and a homology class $\langle f\rangle \in H_1(X)$.

Let Π denote the **abelianisation** of the fundamental group $\pi_1(X, x)$, that is

$$\Pi := \frac{\pi_1(X, x)}{\langle \alpha\beta\alpha^{-1}\beta^{-1} \,:\, \alpha, \beta \in \pi_1(X, x)\rangle}.$$

The result we aim to prove is that Π is isomorphic to the first homology group $\pi_1(X)$. In order to do this however we will need several preliminary techincal lemmas.

Lemma 11. *If f and g are paths in X such that $f(1) = g(0)$ then the 1-chain*

$$c = f \cdot g - f - g$$

is a boundary.

Proof. On the standard 2-simplex Δ_2 put[11] f on the edge (P_0, P_1) and g on the edge (P_1, P_2). Then define a singular 2-simplex $\sigma : \Delta_2 \to X$ to be constant on the lines perpendicular to the edge (P_0, P_2). Then $f \cdot g$ is on the edge (P_0, P_2). Then

$$d\sigma = g - (f \cdot g) - f = c.$$

The point of the previous lemma is to allow us to replace the 1-simplex $f \cdot g$ by the 1-chain $f + g$ modulo boundaries.

Lemma 12. *If f is a path in X then $f + \bar{f}$ is a boundary. A constant path is a boundary.*

Proof. The boundary of a constant 2-simplex is a constant 1-simplex since two of the faces cancel. If we put f on the edge (P_0, P_1) and then define a 2-simplex σ by making it constant on lines parallel to the edge (P_0, P_2), then the edge (P_1, P_2) carries $\bar{f}$ and we have $d\sigma = f + \bar{f} -$ constant. Since the constant edge is a boundary, so is $f + \bar{f}$.

This last lemma allows us to obtain a well defined function $h : \pi_1(X, x) \to H_1(X)$ by

$$h([f]) = \langle f \rangle .$$

In fact, we claim that h is a group homomorphism. To see this, note that by Lemma 11 we have

$$\begin{aligned} h([f \cdot g]) &= h([f \cdot g]) \\ &= \langle f \cdot g \rangle \\ &= \langle f \rangle + \langle g \rangle \\ &= h([f]) + h([g]) . \end{aligned}$$

Thus h factors through the commutator subgroup $\left\langle \alpha\beta\alpha^{-1}\beta^{-1} \,:\, \alpha, \beta \in \pi_1(X, x) \right\rangle$ of $\pi_1(X, x)$ and defines a map $\boldsymbol{h} : \Pi \to H_1(X)$.

Here then, is the statement of the main result.

Theorem 11. (the Hurewicz isomorphism)

The map $\boldsymbol{h} : \Pi \to H_1(X)$ is an isomorphism.

[11] The diagram (which will be reproduced in the talk) on p170 of shows that orientation of Δ_2 used here.

Proof. Let us first define the function $\boldsymbol{\eta} : H_1(X) \to \Pi$ that will be the inverse to $\boldsymbol{h}$. For any point $y \neq x \in X$, let f_y be some path from x to y, and let f_x denote the constant path e_x. Now let c be a 1-chain in X. Define

$$\tilde{c} := f_{c(0)} \cdot c \cdot \bar{f}_{c(1)},$$

which is thus a loop at x. Define $\eta(c) = [\tilde{c}] \in \Pi$. This extends to a homomorphism $\eta : C_1(X) \to \Pi$, which is well defined precisely as we are mapping into Π, not $\pi_1(X, x)$. We will now show that η induces a map $\boldsymbol{\eta} : H_1(X) \to \Pi$.

For this, we show that η maps the group $B_1(X)$ onto the identity element of Π. Let $\sigma : \Delta_2 \to X$ be a 2-simplex, and put $\sigma(P_i) = x_i \in X$ for $i = 0, 1, 2$. Let $f^i := \sigma \circ \partial_2^i$ denote the paths in X corresponding to the faces of σ. Then $f^2 \cdot f^0 \cdot \bar{f}^1$ is homotopic to a constant, since σ is convex, and we have

$$\begin{aligned}
\eta(d\sigma) &= \eta\left(f^2 + f^0 - f^1\right) \\
&= \eta\left(f^2\right)\eta\left(f^0\right)\eta\left(f^1\right)^{-1} \\
&= \left[\tilde{f}^2\right]\left[\tilde{f}^0\right]\left[\tilde{f}^1\right]^{-1} \\
&= \left[f_{x_0} \cdot f^2 \cdot \bar{f}_{x_1} \cdot f_{x_1} \cdot f^0 \cdot \bar{f}_{x_2} \cdot \overline{\left(f_{x_0} \cdot f^1 \cdot \bar{f}_{x_2}\right)}\right] \\
&= \left[f_{x_0} \cdot f^2 \cdot f^0 \cdot \bar{f}^1 \cdot \bar{f}_{x_0}\right] \\
&= \left[e_{x_0}\right] = 0 \in \Pi.
\end{aligned}$$

Now we show that $\boldsymbol{\eta}\boldsymbol{h} = \mathbb{1}$. Indeed, since f_x was chosen to be the constant path e_x at x, if f is a loop at x we have

$$\boldsymbol{\eta}\boldsymbol{h}([f]) = \boldsymbol{\eta}(\langle f \rangle) = [e_x \cdot f \cdot \bar{e}_x] = [f].$$

Finally we need to show that $\boldsymbol{h}\boldsymbol{\eta} = \mathbb{1}$. For this we first observe that the map $y \mapsto f_y$ takes 0-simplices (points) into 1-simplices and thus extends to a homomorphism $F : C_0(X) \to C_1(X)$, namely

$$F\left(\sum n_i y_i\right) = n_i f_{y_i} \in C_1(X).$$

We now claim the following: if σ is a 1-simplex in X then the class $\boldsymbol{h}(\eta(\sigma))$ is represented by the cycle

$$\sigma + f_{\sigma(0)} - f_{\sigma(1)} =: \sigma - f_{\partial\sigma}.$$

To see this we compute

$$\begin{aligned}
\boldsymbol{h}(\eta(\sigma)) &= \boldsymbol{h}\left(\left[f_{\sigma(0)} \cdot \sigma \cdot \bar{f}_{\sigma(1)}\right]\right) \\
&= \left\langle f_{\sigma(0)} \cdot \sigma \cdot \bar{f}_{\sigma(1)}\right\rangle \\
&= \left\langle f_{\sigma(0)} + \sigma + \bar{f}_{\sigma(1)}\right\rangle \\
&= \left\langle \sigma + f_{\sigma(0)} - f_{\sigma(1)}\right\rangle.
\end{aligned}$$

From this, it is clear that if c is a 1-chain then

$$\boldsymbol{h}(\eta(c)) = \langle c - f_{\partial c} \rangle .$$

In particular, if c is a 1-cycle then $\boldsymbol{h}(\eta(c)) = \langle c \rangle$. This completes the proof of the theorem.

Finally let us mention this theorem has a generalisation to the higher homotopy groups; a special case of this says the following:

Theorem 12. *Let X be a simply connected topolgical space. Then the first nonzero homotopy and homology groups occur in the same dimension and are isomorphic.*

8 The homology of the two-holed torus Σ

We will conclude this talk by computing the homology of the two-holed torus Σ. Since Σ is of course path connected, our previous result guarantees us that $H_0(\Sigma) = \mathbb{Z}$. Clearly $H_n(\Sigma) = 0$ for $n \geq 3$, and since Σ has one 2-dimensional hole we have $H_2(\Sigma) = \mathbb{Z}$. Thus the only interesting group is $H_1(\Sigma)$.

We begin with a non-rigorous argument. It is easy to draw[12] four non-nullhomotopic loops a, b, α, β on Σ. It is visually apparent that none of these loops differ by a common boundary, and thus they all represent distinct homology classes in $H_1(\Sigma)$. Moreover, if we try to draw different 1-cycles on Σ these are all seen to represent one of these four homology classes, or else are homologous to zero. For instance, the loop that goes round the centre disk is clearly nullhomotopic.

Thus we are tempted to claim that $H_1(\Sigma) = \mathbb{Z}^4$, generated by the 1-cycles a, b, α and β. There are several ways that this can be made rigorous. One of which is to use the Mayer-Vietoris sequence alluded to earlier. As mentioned before, that involves different ideas to what we have seen in this talk so far. Instead, therefore, we shall use the Hurewicz theorem proved in the last section, and obtain $H_1(\Sigma)$ from knowlege of $\pi_1(\Sigma)$. To compute $\pi_1(\Sigma)$, we use Van Kampen's theorem. We note that Σ may be represented by an octagon with sides $a, b, a^{-1}, b^{-1}, \alpha, \beta, \alpha^{-1}, \beta^{-1}$ identified[13]. As in the proof of π_1 (Klein bottle), we let V denote a small disc in the cetntre of the octagon, and let U denote $\Sigma \backslash \{x\}$, where $x \in V$. Then V is contractible, $U \cap V$ is a circle, and U contracts onto a space X consisting of four circles joined at a point (a 'double' figure of eight). An arguement very similar to that computing $\pi_1(\mathbf{8})$ yields

$$\pi_1(X, x) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle a, b, \alpha, \beta \rangle ,$$

the free group on generators a, b, α and β.

[12] This will be drawn in class.

[13] See picture.

The circle $U \cap V$ deforms onto X and represents the word $aba^{-1}b^{-1}\alpha\beta\alpha^{-1}\beta^{-1}$ there. It thus follows that

$$\begin{aligned}\pi_1(\Sigma, x) &\cong \pi_1(X, x)/N \\ &= \left\langle a, b, \alpha, \beta \,:\, aba^{-1}b^{-1}\alpha\beta\alpha^{-1}\beta^{-1} = 1 \right\rangle.\end{aligned}$$

Finally, we thus have $H_1(\Sigma)$ equal to the abelianisation of $\pi_1(\Sigma, x)$, that is,

$$H_1(\Sigma) = \mathbb{Z}^4,$$

generated by the cycles a, b, α and β.

Exactly the same argument immediately generalises to prove the following result:

Theorem 13. *Let Σ_g denote the g-holed torus. Then*

$$\pi_1(\Sigma_g) = \left\langle a_1, b_1, \ldots, a_g, b_g \,:\, a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \right\rangle,$$

and

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{2g} & n = 1 \\ 0 & n \neq 0, 1, 2. \end{cases}$$

9 Exercises

We conclude with some problems for the reader[14].

Exercise 1. If a space X retracts onto a subspace A, with $r : X \to A$ the retraction and $i : A \hookrightarrow X$ the inclusion, prove that the induced map $r_* : \pi_1(X) \to \pi_1(A)$ is surjective and $i_* : \pi_1(A) \to \pi_1(X)$ is injective. If A is a deformation retract, prove both r_* and i_* are isomorphisms.

Exercise 2. If X and Y are path connected, prove $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Exercise 3. Let X be a topological space, and $f : S^1 \to X$ be continuous. Prove the following are equivalent.

1. f is nullhomotopic,
2. there exists a continuous map $F : D^2 \to X$ such that $F|_{\partial D^2} = f$,
3. $f_* : \pi_1(S^1) \to \pi_1(X)$ is the trivial homomorphism.

Definition 8. *A **topological group** G is a group such that the multiplication map $m : G \times G \to G$, $(g, h) \mapsto gh$ and the inverse map $i : G \to G$, $g \mapsto g^{-1}$ is continuous.*

[14] If you would like hints and solutions to these, just email me.

Exercise 4. Let 1 denote the identity element of G. Given loops f, g based at 1, define the loop $f \times g$ by

$$f \times g\,(s) := m\,(f\,(s)\,, g\,(s))\,.$$

Prove successively that:

1. if $f_0 \sim f_1$ and $g_0 \sim g_1$ rel ∂I then $f_0 \times g_0 \sim f_1 \times g_1$ rel ∂I,
2. $f \times g \sim f \cdot g$ rel ∂I,
3. $\pi_1\,(G, 1)$ is abelian.

Exercise 5. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $m \neq n$.

Exercise 6. Prove that the disc D^{n+1} does not retract onto the sphere S^n for $n \geq 0$. Use this to prove **Brower's fixed point theorem**: if $\psi : D^{n+1} \to D^{n+1}$ is a continuous map then ψ has a fixed point.

We conclude with one slightly harder exercise.

Exercise 7. Compute $\pi_m\,(S^n)$ for all $m, n \geq 0$.

Thanks for reading!

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