# SPECTRAL GEOMETRY

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Abstract. Brief notes on various parts of Dennis Barden's 2009 Part III course on Spectral Geometry

## 1. The definition of the Laplacian

**Problem 1.** *Give three alternative definitions of the Laplacian acting on the functions on a Riemannian manifold and prove their equivalence.*

Let $(M^d, g)$ be a closed connected orientable Riemannian manifold.

- Let $\omega \in \Omega^d(M)$ denote the volume form defined by the metric and the orientation. Suppose $(x^i)$ are local positively oriented coordinates over $U \subseteq M$. Let $G := [g_{ij}]$. Then define $\omega_U \in \Omega^d(U)$ by
$$\omega_U := \sqrt{\det G} dx^1 \wedge \cdots \wedge dx^d.$$
Then $\omega_U$ is defined globally. Indeed, if $(y^i)$ are local positively oriented coordinates over $V \subseteq M$ with $U \cap V \neq \varnothing$ then we need to show that
$$\omega_U | U \cap V = \omega_V | U \cap V.$$
Write
$$F^i_j = \frac{\partial x^i}{\partial y^j}, \quad H^j_i = \frac{\partial y^j}{\partial x^i};$$
note that $F = H^{-1}$. Then
$$dx^i = F^i_j dy^j, \quad \partial_{x^i} = H^j_i \partial_{y^j}.$$
$$dx^1 \wedge \cdots \wedge dx^d = \det F \cdot dy^1 \wedge \cdots \wedge dy^d.$$
Moreover if $g'_{ij} = \langle \partial_{y^i}, \partial_{y^j} \rangle$ then since
$$g_{ij} = \langle \partial_{x^i}, \partial_{x^j} \rangle = H^k_i H^\ell_j \langle \partial_{y^k}, \partial_{y^\ell} \rangle = H^k_i g'_{k\ell} H^j_\ell$$
we have
$$G = HG'H^{\mathrm{T}},$$
and thus
$$\begin{aligned} \sqrt{\det G} dx^1 \wedge \cdots \wedge dx^d &= \sqrt{\det HG'H^{\mathrm{T}}} \det F dy^1 \wedge \cdots \wedge dy^d \\ &= \sqrt{\det G'} dy^1 \wedge \cdots \wedge dy^d, \end{aligned}$$
since $\det HH^{\mathrm{T}} = \frac{1}{(\det F)^2}$.
- Given $f \in C^\infty(M)$, we have $df \in \Omega^1(M)$ and thus $df^\# \in \mathfrak{X}(M)$. Here $\flat : TM \to T^*M$ and $\# : T^*M \to TM$ denote the 'musical' isomorphisms
$$\flat : v^i \partial_i \mapsto g_{ij} v^i dx^j$$
with inverse
$$\# : f_j dx^j \mapsto g^{ij} f_j \partial_i.$$
- Given $X \in \mathfrak{X}(M)$, we define the *divergence* of $X$ to be the function $\mathrm{div}(X) \in C^\infty(M)$ given implicitly by
$$\mathrm{div}(X)\omega = d(i_X \omega).$$

**Definition.** Given $f \in C^\infty(M)$, define the *Laplacian*

$$\Delta f := \mathrm{div}(df^\#).$$

Thus $\Delta f \in C^\infty(M)$.

**Lemma.** *Suppose* $f \in C^\infty(M)$ *and* $X \in \mathfrak{X}(M)$. *Then*

$$\mathrm{div}(fX) = df(X) + f\mathrm{div}(X).$$

*Proof.* Suppose $x \in M$. We prove the result at $x$. This is sufficient, as both sides are functions, and thus it is sufficient to verify it pointwise. If $X(x) = 0$ then both sides are zero. Otherwise, fix a coordinate neighborhood $U \subseteq M$ with $x \in U$ over which $\omega|U$ is given by $\omega_U$ and such that $X \neq 0$ on $U$. Since $TM$ is trivial over $U$, we may pick a local frame $(X = X_1, X_2, \ldots, X_d)$. Then for any 1-form $\alpha \in \Omega^1(U)$ we have

$$\begin{aligned}(\alpha \wedge i_X\omega_U)(X_1, X_2, \ldots, X_d) &= \alpha(X) i_X\omega_U(X_2, \ldots, X_d)\\ &= (\alpha(X)\omega_U)(X_1, X_2, \ldots, X_d).\end{aligned}$$

Since $(X_i)$ is a local frame, we conclude

$$\alpha \wedge i_X\omega_U = \alpha(X)\omega_U.$$

The result follows by applying this to $\alpha = df$, and using the fact that $\omega_U(x) \neq 0$. □

- The metric defines an $L^2$-inner product $\langle \cdot, \cdot \rangle_{L^2}$ on $C^\infty(M)$ by setting

  $$\langle f, h \rangle_{L^2} := \int_M fg\,\omega.$$

  Similarly we obtain an $L^2$-inner product (also denoted by) $\langle \cdot, \cdot \rangle_{L^2}$ on $\Omega^1(M)$ by setting

  $$\langle \alpha, \beta \rangle_{L^2} := \int_M \left\langle \alpha^\#, \beta^\# \right\rangle \omega.$$

- The *adjoint* $\delta$ of the exterior derivative defines (in particular) a map $\Omega^1(M) \to C^\infty(M)$ according to the recipe

  $$\langle f, \delta\alpha \rangle_{L^2} = \langle df, \alpha \rangle_{L^2}.$$

**Definition.** Given $f \in C^\infty(M)$, define the *Laplacian*

$$\Delta f := -\delta df.$$

Thus $\Delta f \in C^\infty(M)$.

**Lemma.** *It holds that*

$$\delta\alpha = -\mathrm{div}(\alpha^\#).$$

*Proof.* We show

$$\langle df, \alpha \rangle_{L^2} = \left\langle f, -\mathrm{div}(\alpha^\#) \right\rangle_{L^2}.$$

For this one observes that

$$df(\alpha^\#) + f\mathrm{div}(\alpha^\#) = \mathrm{div}(f\alpha^\#)$$

by the previous lemma and hence

$$\langle df, \alpha \rangle_{L^2} - \left\langle f, -\mathrm{div}(\alpha^\#) \right\rangle_{L^2} = \int_M \mathrm{div}(f\alpha^\#)\omega = \int_M f d_{\alpha^\#}\omega,$$

and the last integral is zero by Stokes' theorem. □

**Corollary.** *Definition* 1 *is equivalent to Definition* 2.

- Let $f \in C^\infty(M)$. We define the *Hessian* $\mathrm{Hess}(f)$ of $f$ to be $\nabla(df)$; here $\nabla$ denotes the Levi-Civita connection of $(M,g)$ acting on 1-forms. Thus given vector fields $X, Y \in \mathfrak{X}(M)$ we have
$$\mathrm{Hess}(f)(X,Y) := \nabla_X(df)(Y) = X(df(Y)) - df(\nabla_X Y).$$
Suppose $(x^i)$ are *normal* coordinates at a point $x \in M$. Let us compute the expression for $\mathrm{Hess}(f)(x)$ in these coordinates. We have
$$df = \partial_i f dx^i,$$
and hence if
$$\mathrm{Hess}(f) = H_{ij} dx^i \otimes dx^j$$
we have
$$\mathrm{Hess}(f)(\partial_i, \partial_j) = \partial_{ij} f - df(\nabla_{\partial_i}\partial_j),$$
and using
$$\nabla_{\partial_i}\partial_j(x) = 0$$
we have
$$\mathrm{Hess}(f)(x) = \partial_{ij} f(x),$$
that is
$$H_{ij}(x) = \partial_{ij} f(x).$$
- Given a $(0,2)$-tensor $\beta$, we define the *trace* of $\beta$ to be function $\mathrm{tr}(\beta) \in C^\infty(M)$ defined as follows. Suppose $x \in M$. Let $(X_i)$ be an orthonormal frame of $TM$ on a neighborhood of $M$. Then
$$\mathrm{tr}(\beta)(x) := \sum_{i=1}^d \beta(X_i, X_i).$$
- In particular, taking the orthonormal frame to be $(\partial_i)$ where the $(x^i)$ are orthonormal coordinates we have
$$\mathrm{tr}(\mathrm{Hess}(f))(x) = \sum_{i=1}^d \partial_{ii} f(x).$$

**Definition.** Given $f \in C^\infty(M)$, define the *Laplacian*
$$\Delta f := \mathrm{tr}(\mathrm{Hess}(f)).$$
Thus $\Delta f \in C^\infty(M)$.

**Lemma.** *Using Definition 1, the Laplacian is given in local coordinates* $(x^i)$ *on* $U \subseteq M$ *by*
$$\Delta f = g^{ij}\partial_{ij} f + \text{lower order terms}.$$

*Proof.* For convenience, put $\nu := \sqrt{\det G}$ where $G = [g_{ij}]$ so that
$$\omega_U = \nu dx^1 \wedge \cdots \wedge dx^d.$$
Then if $X = a^i \partial_i$ is any vector field we have
$$\begin{aligned} i_X\omega_U(\partial_1, \ldots, \hat{\partial}_i, \ldots, \partial_d) &= \omega_U(X, \partial_1, \ldots, \hat{\partial}_i, \ldots, \partial_d) \\ &= (-1)^{i-1}\omega_U(\partial_1, \ldots, \partial_{i-1}, X, \partial_{i+1}, \ldots \partial_d), \end{aligned}$$
and the last term is equal to $(-1)^{i-1}\nu a^i$, since all the other terms die. Since $\{dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^d\}$ forms a local frame for $\Omega^{d-1}(M)$ we conclude that
$$i_X\omega_U = \sum_{i=1}^d (-1)^{i-1}\nu a^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^d,$$

and thus

$$\begin{aligned} d(i_X\omega_U) &= \sum_{i,j=1}^{d}(-1)^{i-1}\partial_j(\nu a^i)dx^j\wedge dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^d \\ &= \partial_i(\nu a^i)\frac{1}{\nu}\omega_U. \end{aligned}$$

Thus

$$\operatorname{div}(X)=\frac{1}{\nu}\partial_i(\nu a^i).$$

Now take $X = df^{\#}$ to obtain

$$\Delta f=\operatorname{div}(df^{\#})=\frac{1}{\nu}\partial_i(\nu g^{ij}\partial_j f)=g^{ij}\partial_{ij}f+\text{lower order terms}.$$

□

**Corollary.** *Definition* 1 *is equivalent to Definition* 3.

*Proof.* Apply the previous lemma in the special case where the $(x^i)$ are normal coordinates at a point $x\in M$, so $g^{ij}(x)=\delta^{ij}(x)$, and hence in this case the lemma simplifies to give

$$\operatorname{div}(df^{\#})(x)=\sum_{i=1}^{d}\partial_{ii}f(x),$$

which is also equal to $\operatorname{tr}(\operatorname{Hess}(f))(x)$ in these coordinates. □

## 2. The heat kernel

**Problem 2.** *Define the heat kernel, heat trace and heat invariants of a Riemannian manifold. Prove that a heat kernel exists and is unique. Derive an expression for the heat trace involving the spectrum of the Laplacian, and deduce that isospectral manifolds have the same dimension and volume.*

Let $(M^d,g)$ be a closed connected oriented manifold.

- A heat kernel for $(M,g)$ is a function

$$K: M\times M\times\mathbb{R}^+\to\mathbb{R};$$

$$(x,y,t)\mapsto K(x,y,t),$$

  where $\mathbb{R}^+ := \{t\in\mathbb{R} : t>0\}$ with the following properties:
  (1) The map $x\mapsto K(x,y,t)$ is of class $C^0$ for fixed $(y,t)\in M\times\mathbb{R}^+$. The map $y\mapsto K(x,y,t)$ is of class $C^2$ for fixed $(x,t)\in M\times\mathbb{R}^+$. The map $t\mapsto K(x,y,t)$ is of class $C^1$ for fixed $(x,y)\in M\times M$.
  (2) $(\partial_t-\Delta_y)[K]\equiv 0$.
  (3) For any $x\in M$ and any $f\in C^\infty(M)$ it holds that

$$f(x)=\lim_{t\downarrow 0}\int_M K(x,y,t)f(y)\omega_g(y).$$

- Let $L^2(M)$ denote the space of measurable functions $f: M\to\mathbb{R}$ for with finite $L^2$-norm. Then $(L^2(M),\langle\cdot,\cdot\rangle_{L^2})$ is a Hilbert space. The Laplacian $\Delta$ is an elliptic self-adjoint negative-definite operator on $L^2(M)$, and as such its spectrum is discrete and accumulates only at $-\infty$. From standard facts about elliptic operators we deduce the following: each eigenspace of $\Delta$ is finite-dimensional, and any eigenfunction is necessarily of class $C^\infty$. Eigenspaces belonging to distinct eigenvalues are $L^2$-orthogonal. Moreover $L^2(M)$ is the Hilbert sum of the eigenspaces.

- In particular, there exists functions $\{\phi_i\} \subseteq C^\infty(M)$ and $\{\lambda_i\} \subseteq \mathbb{R}^-$ such that

$$\Delta\phi_i = \lambda_i\phi_i$$

and such that

$$\delta_{ij} = \langle\phi_i, \phi_j\rangle_{L^2} = \int_M \phi_i(x)\phi_j(x)\omega_g(x),$$

and such that if $f \in L^2(M)$ and

$$f_i := \langle f, \phi_i\rangle_{L^2}$$

then $\sum_i f_i\phi_i$ converges and is equal to $f$ *as an element of* $L^2(M)$.

**Proposition.** *Assume that a heat kernel $K$ exists for $(M, g)$. Then $K$ is unique.*

*Proof.* If $\{\phi_i\}$ denotes the orthonormal basis of $L^2(M)$ then if

$$f_i(x,t) := \int_M K(x,y,t)\phi_i(y)\omega_g(y)$$

then $\sum_i f_i(x,t)\phi_i(y)$ converges and is equal to $K$ as elements of $L^2(M)$. Since $t \mapsto K(x,y,t)$ is of class $C^1$ and $M$ is compact, we may differentiate under the integral sign to obtain:

$$\begin{aligned}
\partial_t f_i(x,t) &= \int_M \partial_t K(x,y,t)\phi_i(y)\omega_g(y) \\
&= \int_M \Delta_y K(x,y,t)\phi_i(y)\omega_g(y) \\
&= \int_M K(x,y,t)\Delta_y\phi_i(y)\omega_g(y) \\
&= \lambda_i \int_M K(x,y,t)\phi_i(y)\omega_g(y) \\
&= \lambda_i f_i(x,t),
\end{aligned}$$

and hence $f_i(x,t) = F_i(x)e^{\lambda_i t}$. Thus $\lim_{t\downarrow 0} f_i(x,t) = F_i(x)$. But by property (3) of the heat kernel, we necessarily have

$$\lim_{t\downarrow 0} f_i(x,t) = \lim_{t\downarrow 0} \int_M K(x,y,t)\phi_i(y)\omega_g(y) = \phi_i(x),$$

and hence $F_i = \phi_i$. We have shown that if $K_{xt}(y) := K(x,y,t)$ then for all $(x,t) \in M \times \mathbb{R}^+$, the function $\sum_i e^{\lambda_i t}\phi_i(x)\phi_i$ is equal to $K_{xt}$ *as elements of* $L^2(M)$. This implies that for any $(x,t) \in M \times \mathbb{R}^+$ we have as $n \to \infty$ that

$$\left\| K_{xt} - \sum_{i=0}^{n} e^{\lambda_i t}\phi_i(x)\phi_i \right\|_{L^2} \to 0.$$

Hence there exists a subsequence $\{i(k)\} \subseteq \mathbb{N}$ such that for *all* $(x,t) \in M \times \mathbb{R}^+$, the function

$$\sum_{i=0}^{i(k)} e^{\lambda_i t}\phi_i(x)\phi_i$$

converges almost everywhere to $K_{xt}$. We are almost done. Observe that by Parseval's inequality, for all $(x,y,t) \in M \times M \times \mathbb{R}^+$ we have

$$\left\langle K_{x\frac{t}{2}}, K_{y\frac{t}{2}} \right\rangle_{L^2} = \sum_i e^{\lambda_i t}\phi_i(x)\phi_i(y).$$

In particular $\sum_i e^{\lambda_i t}\phi_i(x)\phi_i(y)$ converges for all $(x,y,t) \in M \times M \times \mathbb{R}^+$ as is continuous as a function of all three variables (since the same is true of $\left\langle K_{x\frac{t}{2}}, K_{y\frac{t}{2}} \right\rangle_{L^2}$. This completes the proof. ∎

- Let $x \in M$ and let $\varepsilon > 0$ be such that $\exp_x : D_\varepsilon(x) \to B_\varepsilon(x)$ is a diffeomorphism. Here
$$D_\varepsilon(x) := \{v \in T_xM \,:\, \|v\|_x < \varepsilon\}$$
and
$$B_\varepsilon(x) := \{y \in M \,:\, d(x,y) < \varepsilon\}\,.$$
Choose coordinates $(r, \varphi^1, \ldots, \varphi^{d-1})$ on $D_\varepsilon(x)$, where $(\varphi^1, \ldots, \varphi^{d-1})$ are coordinates on the sphere $S_xM := \{v \in T_xM \,:\, \|v\|_x = 1\}$ and $r$ is the radial coordinate. Via $\exp_x^{-1}$, we may consider these coordinates as defining coordinates $(r, \varphi^1, \ldots, \varphi^{d-1})$ on $B_\varepsilon(x)$ (such coordinates are then called *geodesic polar coordinates*). Let $G$ denote the matrix given by the metric in this chart. Define
$$\nu : B_\varepsilon(x) \to \mathbb{R};$$
$$\nu(r,\varphi) = \sqrt{\det G(r,\varphi)}.$$
Suppose now $f \in C^\infty(B_\varepsilon(x))$ is a radial function, that is there exists $\psi : \mathbb{R} \to \mathbb{R}$ such that,
$$f(r,\varphi) = \psi(r).$$
Then we can compute the Laplacian $\Delta f$ in these coordinates by:
$$\Delta f = \partial_{rr}\psi + \partial_r\psi\left(\frac{\nu'}{\nu} + \frac{d-1}{r}\right). \tag{2.1}$$
- Let $f, h \in C^\infty(M)$. Then
$$\Delta(fh) = f\Delta h + 2\langle df, dg\rangle + h\Delta f. \tag{2.2}$$
Indeed, if $\alpha$ is any 1-form then
$$\delta(f\alpha) = -\mathrm{div}(f \cdot \alpha^\#) = df(\alpha^\#) + f\mathrm{div}(\alpha^\#),$$
and thus
$$\delta(fdh) = f\delta dh - \langle df, dh\rangle\,.$$
Hence
$$\begin{aligned}\Delta(fh) &= -\delta d(fh)\\ &= -\delta(fdh + hdf)\\ &= -f\delta dh + 2\langle df, dh\rangle - h\delta df\\ &= f\Delta h + 2\langle df, dh\rangle + h\Delta f.\end{aligned}$$

**Theorem.** *There exists a heat kernel on* $(M,g)$.

*Proof.* Since $M$ is closed, the *convexity radius* $\rho$ of $(M,g)$ is positive. By definition, the convexity radius of $(M,g)$ is defined by
$$\rho = \inf_{x\in M} \rho_x;$$
$\rho_x := \sup\{r > 0 \,:\, \text{any two points } y, z \in B_r(x) \text{ are joined by a unique geodesic segment}\}$.

Set
$$W := \{(x,y) \in M \,:\, d(x,y) < \rho\}.$$
We will define the heat kernel on $W$ in several stages. First define $G : U \times \mathbb{R}^+ \to \mathbb{R}$ by
$$G(x,y,t) := \frac{1}{(4\pi t)^{d/2}} e^{-\frac{d^2(x,y)}{4t}}.$$

By choice of $U$, $d^2|U$ is of class $C^\infty$ and hence so is $G$. Since $G$ is radial, by (2.1) we have

$$\begin{aligned}\Delta_y G &= -\partial_{rr}G - \partial_r G\left(\frac{\nu'}{\nu} + \frac{d-1}{r}\right)\\ &= -\left(d + \frac{r\nu'}{\nu} - \frac{r^2}{2t}\right)\frac{G}{2t},\end{aligned}$$

using the fact that

$$\partial_r G = -\frac{r}{2t}G, \quad \partial_{rr}G = -\frac{1}{2t}\left(1 - \frac{r^2}{2t}\right)G.$$

We must modify $G$ to obtain the heat kernel we want, and we shall do so as follows: we find functions $\{u_i : W \to \mathbb{R}\}$ such that if $U_k : W \times \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$U_k(x,y,t) := \sum_{i=0}^{k} u_i(x,y)t^i$$

and $S_k : W \times \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$S_k(x,y,t) := G(x,y,t)U_k(x,y,t)$$

then for $k$ sufficiently large, $S_k$ will (almost) be a heat kernel.

Using (2.2), we have

$$\Delta_y S_k = U_k \Delta G + 2\langle dG, dU_k\rangle + G\Delta_y U_k,$$

In geodesic polar coordinates

$$dG = \partial_r G \cdot \partial_r + \partial_{\varphi^i} G \cdot \partial_{\varphi^i}.$$

Since $G$ is radial, $\partial_{\varphi^i}G = 0$ and thus $dG = \partial_r G \cdot \partial_r$. The Gauss lemma tells us that

$$\langle \partial_r, \partial_r\rangle = 1, \quad \langle \partial_r, \partial_{\varphi^i}\rangle = 0,$$

and hence

$$\langle dG, dU_k\rangle = \partial_r G \cdot \partial_r U_k.$$

From this we conclude that

$$(\partial_t - \Delta_y)[S_k] = G\left(\frac{\nu' r}{2\nu t}U_k + \frac{r}{t}\partial_r U_k + \partial_t U_k - \Delta_y U_k\right).$$

Our aim is to choose the $\{u_i\}$ such that for each $k$ we have

$$\frac{\nu' r}{2\nu t}U_k + \frac{r}{t}\partial_r U_k + \partial_t U_k - \Delta_y U_k = \Delta_y u_k \cdot t^k. \tag{2.3}$$

This is the best we can do; ideally we would simply choose the $\{u_i\}$ such that

$$\frac{\nu' r}{2\nu t}U_k + \frac{r}{t}\partial_r U_k + \partial_t U_k - \Delta_y U_k = 0$$

but this is in general not possible. Assuming (2.3) for a second, we briefly sketch how this leads to the construction of the heat kernel. Pick a bump function $\eta \in C^\infty(M \times M)$ such that $\eta \equiv 1$ on $\{(x,y) \in M \times M : d(x,y) < \rho/2\}$ and $\eta \equiv 0$ on $M \times M \backslash W$. Set $H_k := \eta S_k$. Then $H_k \in C^\infty(M \times M \times \mathbb{R}^+)$. Then it turns out that if $k > d/2$ then $H_k$ satisfies properties (1) and (3) of the heat kernel. In order to make $H_k$ satisfy property (2) one needs to take a sum with signs of convolutions of $H_k$; this part is non-examinable and so is omitted.

It remains therefore to verify (2.3). To do this we induct on $k$. The coefficient of $t^{-1}$ is

$$\frac{r\nu'}{2\nu}u_0 + r\partial_r u_0$$

and the coefficient of $t^i$ for $0 \le i \le k-1$ is

$$\left(\frac{r\nu'}{2\nu} + i + 1\right) u_{i+1} + r\partial_r u_{i+1} - \Delta_y u_i.$$

We conclude that

$$\partial_r(\log u_0) = -\frac{1}{2}\partial_r(\log \nu),$$

from which it follows that $u_0$ is (up to a constant) given by

$$u_0 = \frac{1}{\sqrt{\nu}}. \tag{2.4}$$

Now make the lucky guess that $u_i$ is of the form

$$u_i = \frac{k(r)}{r^i\sqrt{\nu}}$$

for some function $k(r)$. Simple manipulations tell us that this choice of $u_i$ will work if and only if the function $k$ satisfies

$$\partial_r k = r^{i-1}\sqrt{\nu} \cdot \Delta_y u_{i-1}.$$

Let $\{x(s) : s \in [0, r]\}$ be the unique unit speed geodesic running from $x$ to $y$. Assuming we have already defined $u_{i-1}$ and using the fact that $\Delta_y u_{i-1}$ is a function of $r$ along this geodesic, we can solve this equation to get

$$k(r) = \int_0^r \sqrt{\nu(x(s))} \cdot \Delta_y u_{i-1}(x(s), y) \cdot s^{i-1} ds;$$

putting this altogether gives

$$u_i(x, y) = \frac{1}{(d(x, y))^i\sqrt{\nu}} \int_0^r \sqrt{\nu(x(s))} \Delta_y u_{i-1}(x(s), y) \cdot s^{i-1} ds,$$

and this completes the verification of (2.3). □

- The *trace of the heat kernel* is the function $Z : \mathbb{R}^+ \to \mathbb{R}$ given by

  $$Z(t) := \int_M K(x, x, t)\omega_g(x),$$

  where $K$ is the heat kernel on $(M, g)$. Using the proposition that

  $$K(x, y, t) = \sum_i e^{\lambda_i t}\phi_i(x)\phi_i(y)$$

  we can easily prove that

  $$Z(t) = \sum_i e^{\lambda_i t}.$$

  Indeed, since $\sum_i e^{\lambda_i t}\phi_i(x)^2$ converges to $K(x, x, t)$ for all $x$ and each term is non-negative, the dominated convergence theorem tells us we can integrate term by term to deduce

  $$\int_M K(x, x, t)\omega_g(x) = \sum_i e^{\lambda_i t} \langle \phi_i, \phi_i \rangle_{L^2} = \sum_i e^{\lambda_i t}.$$

**Corollary.** *The trace of the heat kernel determines the eigenvalues of the Laplacian.*

*Proof.* Since $M$ is closed, the only harmonic functions are constant; thus $\lambda_0 = 0$ and $\lambda_1 < 0$ (where we order the eigenvalues of $\Delta$ as $\lambda_0 > \lambda_1 > \lambda_2 \ldots$, counted with multiplicities. Assuming we already know $\lambda_0, \lambda_1, \ldots, \lambda_i$ for $i \ge 0$ we have

$$\lambda_{i+1} = \min\left\{\mu < 0 : \lim_{t\to\infty} \frac{Z(t) - \sum_{j=0}^i e^{\lambda_j t}}{e^{\mu t}} < \infty\right\}.$$

□

- A function $A:\mathbb{R}^+\to\mathbb{R}$ has *asymptotic expansion* $A(t)\overset{t\downarrow 0}{\approx}\sum_i b_i t^i$ as $t\downarrow 0$ if
$$\lim_{t\downarrow 0}\frac{A(t)-\sum_{i=0}^N b_i t^i}{t^N}=0$$
for all $N\in\mathbb{N}$.
- The function $K(x,x,t)$ has asymptotic expansion
$$K(x,x,t)\overset{t\downarrow 0}{\approx}\frac{1}{(4\pi t)^{d/2}}\sum_i u_i(x,x)t^i.$$
All that remains to show this is to check the required convergence, and this is omitted due to the fact that this uses details not given in the main proof. Integrating term by term (justified as above) however we obtain
$$Z(t)\overset{t\downarrow 0}{\approx}\frac{1}{(4\pi t)^{d/2}}\sum a_i t^i$$
where
$$a_i:=\int_M u_i(x,x)\omega_g(x).$$

**Corollary.** *If $(M,g)$ and $(M',g')$ are isospectral then they have the same volume and the same dimension.*

*Proof.* Isospectrality implies $Z_{(M,g)}(t)=Z_{(M',g')}(t)$ for all $t$. In particular, $d=d'$ and $a_0(M,g)=a_0(M',g')$. But from (2.4) we have
$$a_0=\int_M u_0(x,x)\omega_g(x)=\int_M\frac{1}{\sqrt{\nu(0)}}\omega_g(x)=\int_M\omega_g(x)=\mathrm{vol}(M,g).$$
□

## 3. Sunada's Theorem

**Problem 3.** *Prove Sunada's Theorem: if $N$ is a Riemannian covering of $M_1$ and $M_2$ with covering transformation groups $U_1$ and $U_2$ that are almost conjugate in a group $T$ of isometries of $N$, then $M_1$ and $M_2$ are isospectral.*

*Prove that, if $N$ is the universal covering of $M_0$ with $\pi_1(M_0)=T$ and if $U_1$ and $U_2$ are almost conjugate subgroups of $T$, then the quotient spaces $M_i:=U_i\backslash N$ are isospectral provided the metrics are chosen so that all the coverings are local isometries.*

*Explain briefly how given a Sunada triple $(T,U_1,U_2)$ we can construct a pair of isospectral non-isometric manifolds.*

- Let $p:N\to M$ be a covering map between two smooth manifolds. Suppose $g$ is a Riemannian metric on $M$. Then $p^*g$ is a Riemannian metric on $N$, and if we equip $N$ with this metric then $p$ is locally isometric. We call $p:(N,p^*g)\to(M,g)$ a *Riemannian covering.* We say $p$ is of order $k$ if $\left|p^{-1}(x)\right|=k$ for all $x\in M$. If in addition $p:M\to N$ is a *normal* covering then if $p$ is of order $k$ then the group $T$ of deck transformations consists of precisely $k$ elements $\phi_1,\dots,\phi_k$.
- From now on assume that $M$ is *complete*; this is always the case if $M$ is compact and means the following. Given any two points $x,y\in M$ there exists a unique distance minimizing geodesic running from $x$ to $y$. Moreover every geodesic is defined for all $t\in\mathbb{R}$. Consider $SM$, the unit sphere $SM:=\{(x,v)\in TM : |v|_x=1\}$. Then for any $v\in S_xM$ we define the unique geodesic $\gamma_{(x,v)}$ *adapted* to $(x,v)$ by
$$\gamma_{(x,v)}(t):=\exp_x(tv).$$

In other words, $\gamma_{(x,v)}$ is the unique geodesic such that $\gamma_{(x,v)}(0) = x$ and $\dot{\gamma}_{(x,v)}(0) = v$. Since $M$ is complete, $\gamma_{(x,v)}(t)$ is defined for all $t \in \mathbb{R}$. However it is only *distance minimizing* for $t$ sufficiently small. Thus there exists $\tau(x,v) \in \mathbb{R}^+ \cup \{+\infty\}$ such that $\gamma_{(x,v)}(t)$ is length minimizing from $t \in [0, \tau(x,v)]$, but is not length minimizing for $t > \tau(x,v)$. If $M$ is compact, then $\tau(x,v) < \infty$ for all $(x,v) \in SM$.

- Fix $x \in M$. We define the *cut locus* of $x$, written $C(x)$ to be the set
$$C(x) := \{\exp_x(\tau(x,v)v) : v \in S_x M\}.$$
It is a non-trivial theorem in Riemannian geometry that for any $x \in M$, $C(x)$ has measure zero, and $M \backslash C(x)$ is diffeomorphic to a ball $B \subseteq \mathbb{R}^d$.
- In particular, when integrating over $M$ it makes no difference if we integrate over the cut locus $C(x)$:
$$\int_M f\omega_g = \int_{C(x)} f\omega_g \quad \text{for all } x \in M, f \in C^\infty(M).$$
In the case of a Riemannian covering $p : (N, p^*g) \to (M,g)$ of order $k$, since $M\backslash C(x)$ is diffeomorphic to a ball, $p|p^{-1}(M\backslash C(x))$ is necessarily a trivial covering, that is, $p^{-1}(M\backslash C(x))$ is diffeomorphic to $k$ disjoint balls. In other words, we can write
$$p^{-1}(M\backslash C(x)) = N_1 \sqcup \cdots \sqcup N_k,$$
with $p_i := p|N_i : N_i \to M\backslash C(x)$ an isometry.
- Given $y \in M\backslash C(x)$, let $\widetilde{y}_i \in N_i$ be the unique point in $N_i$ such that $p(\widetilde{y}_i) = y_i$. Thus $p_i$ is map $\widetilde{y}_i \mapsto y$. Then if $T = \{\phi_1 = \mathrm{Id}, \phi_2, \dots, \phi_k\}$ we may assume that $\phi_i(\widetilde{y}_1) = \widetilde{y}_i$ for all $y \in M\backslash C(x)$. In other words, $p_i \circ \phi_i = p_1$.
- As $y$ ranges over $M\backslash C(x_0)$, $\widetilde{y}_i := p_i^{-1}(y)$ ranges over $N_i$. In particular, by definition of the integral we have for any $f \in C^\infty(M)$ that
$$\int_M f(y)\omega_g(y) = \int_{M\backslash C(x)} f(y)\omega_g(y) = \int_{N_i} (p_i^* f)(\widetilde{y}_i)\omega_{p^*g}(\widetilde{y}_i),$$
where $p_i^* f$ is the function $f \circ p_i$.

**Proposition.** *Let $p : (N, p^*g) \to (M,g)$ be a finite normal Riemannian covering and $T$ denote the group of deck transformations of the covering $N \to M$. Let $K_N$ denote the heat kernel on $N$. Then the heat kernel on $M$ is given by*
$$K_M(x,y,t) := \sum_{\phi \in T} K_N(\widetilde{x}, \phi(\widetilde{y}), t),$$
*where $\widetilde{x} \in p^{-1}(x)$ and $\widetilde{y} \in p^{-1}(y)$ are arbitrary.*

*Proof.* The first thing to do is check that the expression for $K_M$ is well defined. Pick an arbitrary point $x_0 \in M$, and write $p^{-1}(M\backslash C(x_0)) = N_1 \sqcup \cdots \sqcup N_k$ as above. Given $x \in M\backslash C(x_0)$, let $\widetilde{x}_i$ denote the unique point in $N_i$ such that $p_i = p|N_i$ satisfies $p_i(\widetilde{x}_i) = x$. Write $T = \{\phi_1 = \mathrm{Id}, \dots, \phi_k\}$ where $\phi_i$ maps $\widetilde{x} \mapsto \widetilde{x}_i$. It suffices to show that for all $(x,y,t) \in (M\backslash C(x_0)) \times (M\backslash C(x_0)) \times \mathbb{R}^+$ and all $i, \ell = 1, \dots, k$ we have
$$\sum_{j=1}^k K_N(\widetilde{x}_1, \phi_j(\widetilde{y}_1), t) = \sum_{j=1}^k K_N(\widetilde{x}_i, \phi_j(\widetilde{y}_\ell), t).$$
For this we need to know the following: for any isometry $\psi : N \to N$, it holds that
$$K_N(\psi(\widetilde{x}), \psi(\widetilde{y}), t) = K_N(\widetilde{x}, \widetilde{y}, t) \quad \text{for all } (\widetilde{x}, \widetilde{y}, t) \in N \times N \times \mathbb{R}^+. \tag{3.1}$$

This holds since the Laplacian, and thus also the heat operator, are invariant under isometries. Thus:

$$\begin{aligned}\sum_{j=1}^{k} K_N(\widetilde{x}_i, \phi_j(\widetilde{y}_\ell), t) &= \sum_{j=1}^{k} K_N(\phi_i^{-1}(\widetilde{x}_1), \phi_j\phi_\ell(\widetilde{y}_1), t) \\ &= \sum_{j=1}^{k} K_N(\widetilde{x}_1, \phi_i^{-1}\phi_j\phi_\ell(\widetilde{y}_1), t) \\ &= \sum_{m=1}^{k} K_N(\widetilde{x}_1, \phi_{m(i,j,\ell)}(\widetilde{y}_1), t),\end{aligned}$$

where $m(i,j,\ell)$ is the unique number in $\{1,\dots,k\}$ such that $\phi_{m(i,j,\ell)} = \phi_i^{-1}\phi_j\phi_\ell$. The point is that since $T$ acts strictly transitively, for any fixed $j,\ell$ the set $\{m(i,j,\ell) : j = 1,\dots,k\}$ is precisely the set $\{1,\dots,k\}$.

Now we check that $K_M$ is a heat kernel on $(M,g)$. Properties (1) and (2) are immediate, since they are both local assertions that are preserved under sum, and $p$ is a local isometry.

The third condition however needs some work. Choose $(x,t) \in M \times \mathbb{R}^+$ and $f \in C^\infty(M)$. We want to show that

$$\int_M K_M(x,y,t) f(y)\omega_g(y) = f(x).$$

We have:

$$\begin{aligned}\int_M K_M(x,y,t)f(y)\omega_g(y) &= \int_{M\setminus C(x_0)} K_M(x,y,t)f(y)\omega_g(y) \\ &= \sum_{i=1}^{k}\int_{M\setminus C(x_0)} K_N(\widetilde{x}_1, \phi_i(\widetilde{y}_1), t)f(y)\omega_g(y) \\ &= \sum_{i=1}^{k}\int_{N_i} K_N(\widetilde{x}_1, \widetilde{y}_i, t)(p_i^* f)(\widetilde{y}_i)\omega_{p^*g}(\widetilde{y}_i) \\ &= \int_{N_1\sqcup\cdots\sqcup N_k} K_N(\widetilde{x}_1, \widetilde{y}, t)(f\circ p)(\widetilde{y})\omega_{p^*g}(\widetilde{y}) \\ &= \int_N K_N(\widetilde{x}_1, \widetilde{y}, t)(f\circ p)(\widetilde{y})\omega_{p^*g}(\widetilde{y}) \\ &= (f\circ p)(\widetilde{x}_1) \\ &= f(x).\end{aligned}$$

This completes the proof. □

**Lemma.** *Let $(N,h)$ be a Riemannian manifold, and let $K_N : N \times N \times \mathbb{R}^+ \to \mathbb{R}$ denote the heat kernel of $(N,h)$. Suppose $\psi \in \mathrm{Isom}(N,h)$ and $\phi \in \mathrm{Diff}(N)$. Then for all $t \in \mathbb{R}^+$, it holds that*

$$\int_N K_N(x, \psi\phi\psi^{-1}(x), t)\omega_h(x) = \int_N (x, \phi(x), t)\omega_h(x).$$

*Proof.* We have

$$\begin{aligned}\int_N K_N(x, \psi\phi\psi^{-1}(x), t)\omega_h(x) &= \int_N K_N(\psi^{-1}(x), \phi\psi^{-1}(x), t)\omega_h(x) \\ &= \int_N K_N(\psi^{-1}(x), \phi\psi^{-1}(x), t)\omega_h(\psi^{-1}(x)) \\ &= \int_N K_N(z, \phi(z), t)\omega_h(z),\end{aligned}$$

where $z = \psi^{-1}(x)$; this works as $\psi$ is an isometry. □

**Corollary.** *Let $p:(N,p^*g)\to(M,g)$ be a finite normal Riemannian covering, and let $T$ denote the group of deck transformations. Then*

$$Z_M(t)=\sum_{[\phi]\subseteq T}\frac{|[\phi]|}{|T|}\int_N K_N(\widetilde{x},\phi(\widetilde{x}),t)\omega_{p^*g}(\widetilde{x}).$$

*Proof.* We have

$$\begin{aligned}
Z_M(t) &= \int_M K_M(x,x,t)\omega_g(x)\\
&= \int_{M\setminus C(x_0)}\sum_{i=1}^k K_N(\widetilde{x}_1,\phi_i(\widetilde{x}_1),t)\omega_g(x)\\
&\overset{(*)}{=} \sum_{i,j=1}^k\frac{1}{k}\int_{M\setminus C(x_0)}K_N(\phi_j(\widetilde{x}_1),\phi_j\phi_i(\widetilde{x}_1),t)\omega_g(x)\\
&= \sum_{i,j=1}^k\frac{1}{k}\int_{N_j}K_N(\widetilde{x}_j,\phi_j\phi_i(\widetilde{x}_1),t)\omega_g(\widetilde{x}_j)\\
&= \sum_{i,j=1}^k\frac{1}{k}\int_{M\setminus C(x_0)}K_N(\widetilde{x}_j,\phi_j\phi_i\phi_j^{-1}(\widetilde{x}_j),t)\omega_g(x)\\
&= \sum_{i,j=1}^k\frac{1}{k}\int_{N_j}K_N(\widetilde{x}_j,\phi_j\phi_i\phi_j^{-1}(\widetilde{x}_j),t)\omega_{p^*g}(\widetilde{x}_j)\\
&\overset{(\dagger)}{=} \sum_{i,j=1}^k\frac{1}{k}\int_{N_j}K_N(\widetilde{x}_j,\phi_i(\widetilde{x}_j),t)\omega_{p^*g}(\widetilde{x}_j)\\
&= \sum_{i=1}^k\frac{1}{k}\int_{N_1\sqcup\cdots\sqcup N_k}K_N(\widetilde{x},\phi_i(\widetilde{x}),t)\omega_{p^*g}(\widetilde{x})\\
&\overset{(\dagger)}{=} \sum_{[\phi]\subseteq T}\frac{|[\phi]|}{|T|}\int_N K_N(\widetilde{x},\phi(\widetilde{x}),t)\omega_{p^*g}(\widetilde{x}),
\end{aligned}$$

where $(*)$ used (3.1) and both the $(\dagger)$'s used the previous lemma. □

**Corollary.** *Let $p:(N,p^*g)\to(M,g)$ be a finite normal Riemannian covering, and let $U$ denote the group of deck transformations. Suppose $U\le T$, where $T\le$ $\mathrm{Isom}(N,p^*g)$. Then*

$$Z_M(t):=\sum_{[\phi]_T\subseteq T}\frac{|[\phi]_T\cap U|}{|U|}\int_N K_N(\widetilde{x},\phi(\widetilde{x}),t)\omega_{p^*g}(\widetilde{x}).$$

*Proof.* The sum is well defined since the finite group $U$ can intersect only finitely many of the conjugacy classes $[\cdot]_T$ of $T$ non-trivially. From the last lemma, if $\phi,\psi\in U$ are conjugate in $T$ then for all $t\in\mathbb{R}^+$, it holds that

$$\int_N K_N(\widetilde{x},\phi(\widetilde{x}),t)\omega_{p^*g}(\widetilde{x})=\int_N K_N(\widetilde{x},\psi(\widetilde{x}),t)\omega_{p^*g}(\widetilde{x}).$$

Thus the corollary is immediate from the previous one, by aggregating the conjugacy classes $[\cdot]_U$ of $U$ that are contained in the same conjugacy class $[\cdot]_T$ of $T$. □

- Recall that if $U_1,U_2\le T$ are finite subgroups such that for all $\phi\in T$,

  $$|[\phi]_T\cap U_1|=|[\phi]_T\cap U_2|$$

  then we say that $U_1$ and $U_2$ are *almost conjugate*. Clearly in this case $|U_1|=|U_2|$. If $U_1$ and $U_2$ are *not* conjugate then we say that $(T,U_1,U_2)$ is a *Sunada triple*. Using the fact that the trace of the heat kernel determines the spectrum of the Laplacian, we immediately obtain *Sunada's theorem.*

**Theorem.** *(Sunada's theorem)*

*Let $p_i : (N,h) \to (M_i, g_i)$ be finite normal Riemannian coverings (i.e. $h = p_i^* g_i$ for $i = 1,2$). Let $U_i$ denote the deck transformation group of $p_i$, and let $T \leq$* Isom$(N,h)$. *Suppose that $U_1$ and $U_2$ are almost conjugate. Then $M_1$ and $M_2$ are isospectral.*

**Lemma.** *Under the hypotheses of Sunada's theorem, if $U_1$ and $U_2$ are conjugate in Isom$(N,h)$ then $M_1$ and $M_2$ are isometric.*

*Proof.* Let $\psi \in \text{Isom}(N,h)$ be an isometry conjugating $M_1$ and $M_2$. Use $\psi$ to define an isometry $\hat{\psi} : (M_1, g_1) \to (M_2, g_2)$ as follows. Suppose $x \in M_1$, and suppose $x = p_1(y)$. Define $\hat{\psi}(x) = p_2\psi(y)$. This is well defined, as if $y' \in p_1^{-1}(x)$ then there exists $\phi \in U_1$ such that $y' = \phi(y)$. Then if $\phi' \in U_2$ is the element $\psi\phi\psi^{-1}$ then

$$p_2\psi(y') = p_2\psi\phi(y) = p_2\phi'\psi(y) = p_2\psi(y)$$

since $p_2\phi' = p_2$.

To see that $\hat{\psi}$ is an isometry, it is enough to observe $\hat{\psi}$ is a bijective local isometry, since then $\hat{\psi}$ is automatically an isometry. First, $\hat{\psi}$ is bijective, as we may define an inverse by applying the same construction to $\psi^{-1} \in \text{Isom}(N,h)$. Secondly, $\hat{\psi}$ is thus an isometry as the same is true of $p_1, p_2$ and $\psi$. □

- A metric $g$ on a manifold $M$ is said to be *bumpy* if the following condition holds for all non-empty open subsets $U, V \subseteq M$: if there exists an isometry between the Riemannian manifolds $(U, i_U^* g)$ and $(V, i_V^* g)$ then necessarily $U = V$ (here $i_U : U \hookrightarrow M$ and $i_V : V \hookrightarrow M$ denote the inclusions).

**Proposition.** *Suppose $p_0 : (N, p_0^* g_0) \to (M_0, g_0)$ is a finite normal Riemannian covering, where $g_0$ is a bumpy metric on $M_0$. Let $T$ denote the group of deck transformations of $p_0$, and let $U_1, U_2 \leq T$. Let $p_i : (M_i, p_i^* g_0) \to (M_0, g_0)$ be the covering induced by the subgroup $U_i$ for $i = 1,2$, that is, $M_i \cong N/U_i$. Then if $M_1$ and $M_2$ are isometric then $U_1$ and $U_2$ are conjugate in $T$.*

*Proof.* Denote by $q_i : M_i \to M_0$ the covering maps. Then by standard covering space theory, $p_0 = p_i \circ q_i$ for $i = 1,2$. Suppose $\psi : (M_1, p_1^* g_0) \to (M_2, p_2^* g_0)$ is an isometry. Then $\psi$ is necessarily fibre preserving, since $g_0$ is a bumpy metric (and so points $x_i \in M_i$ that admit isometric neighborhoods must project to the same point in $M_0$). Standard covering space theory then tells us that the subgroups $G_i := q_{i*}(\pi_1(M_i)) \leq \pi_1(M_0)$ are conjugate. Standard covering space theory tells us that

$$T \cong \pi_1(M_0)/p_{0*}(\pi_1(N));$$

let $\Phi : T \to \pi_1(M_0)/p_{0*}(\pi_1(N))$ denote this map. Then

$$\Phi(U_i) = q_{i*}(\pi_1(M_i)/p_{i*}(\pi_1(N))) \cong q_{i*}(\pi_1(M_i))/p_{0*}(\pi_1(N)).$$

Since $p_0 : N \to M_0$ is a normal covering $p_{0*}(\pi_1(N))$ is a normal subgroup of $\pi_1(M_0)$, the fact that $q_{i*}(\pi_1(M_i))$ are conjugate in $\pi_1(M_0)$ implies that $\Phi(U_i)$ are conjugate in $\pi_1(M_0)/p_{0*}(\pi_1(N))$. Since $\Phi$ is an isomorphism, the $U_i$ are conjugate in $T$. □

**Corollary.** *In the situation described in the previous proposition, a necessary and sufficient condition for $(M_1, p_1^* g_0)$ and $(M_2, p_2^* g_0)$ to be an isospectral non-isometric pair is if $(T, U_1, U_2)$ is a Sunada triple.*

- Suppose we have a Sunada triple $(T, U_1, U_2)$. Then we can use the material above to build isospectral non-isometric manifolds, subject to being able to solve two problems. Firstly, we need to be able to find a manifold $M_0$ with $\pi_1(M_0) \cong T$. Secondly, we need to be able to give $M_0$ a bumpy metric. We won't address the second question, as whilst this is indeed always possible

(*Sunada's lemma* tells us that the set of bumpy metrics is residual in the set of all metrics on a closed manifold), this isn't examinable this year. The first point is probably semi-examinable. The precise statement is: *let $T$ be a finitely presented group. Then for any $d \geq 4$ there exists a closed manifold $M^d$ with $\pi_1(M) \cong T$.* I won't go into any of the details here - I think the exposition in his notes is very good.