# FLOER HOMOLOGY OF COTANGENT BUNDLES

WILL MERRY

Abstract. We consider the cotangent bundle of a closed Riemannian manifold, and show that for Hamiltonians $H : T^*M \to \mathbb{R}$ of the form kinetic plus potential energy we can define the Floer homology $HF_*(T^*M, \omega, H, J)$, where $\omega$ is the canonical symplectic form $dp \wedge dx$ on $T^*M$ and $J$ is an almost complex structure close to the one induced by the Riemannian metric. We follow the approach of Abbondandolo and Schwarz [AS], and construct an isomorphism between the Floer complex $CF_*(T^*M, \omega, H, J)$ and the Morse complex $CM_*(\mathcal{L}M, L, \mathcal{G})$, where $L$ is a Lagrangian on $TM$ related to $H$ by the inverse Legendre transformation, $\mathcal{L}M$ is the free loop space on $M$ and $\mathcal{G}$ is a Morse-Smale metric on $\mathcal{L}M$ for $L$. By the work of Abbondandolo and Majer [AM], the Morse homology $CM_*(\mathcal{L}M, L, \mathcal{G})$ is isomorphic to the singular homology $H_*^{\text{sing}}(\mathcal{L}M)$, and we begin with an exposition of this result. We also show that the same result holds with $\omega$ replaced by a twisted symplectic form $\Omega = \omega + \tau^*\sigma$ for an exact 2-form $\sigma$ on $M$.

## 1. Introduction

Let $(M, g)$ be a closed (i.e. compact and without boundary) orientable Riemannian manifold with fixed metric $g$. The aim of this paper is to summarize the recent paper [AS] by Abbondandolo and Schwarz on the isomorphism between the Floer homology of the cotangent bundle of $M$ and the singular homology of its free loop space. I have made three simplifying assumptions throughout.

Firstly I will assume that $M$ is simply connected; this is mainly to avoid overuse of the word 'contractible' and is not a necessary restriction. In fact, if the loop space has several connected components then one simply defines the various complexes and isomorphisms we will construct separately for each component.

The second is more restrictive. Abbondandolo and Schwarz develop the theory for a class of Hamiltonians satisfying certain growth conditions (essentially that they grow at worst quadratically outside of a compact set). I however have restricted to Hamiltonians of the 'natural' form kinetic plus potential energy; this is the same type of Hamiltonian as was considered in the (entirely different) proof of the same result given by Salamon and Weber in [SW]. These are easily seen to satisfy the growth conditions that Abbondandolo and Schwarz studied. In order to obtain results for a twisted symplectic form (see below) I also need to consider a slight generalisation of this type of Hamiltonian, by adding in an extra term representing the effect of a magnetic field. Again this generalisation clearly satisfies the Abbondandolo and Schwarz growth conditions.

Thirdly, and this is the most severe, I will take $\mathbb{Z}_2$ homology coefficients throughout, thus sidestepping the orientation question. In fact the results all hold for coefficients in any prinicipal ideal domain; see [AS]. That said, the proof I give of the Morse Homology Theorem is via a cellular filtration argument, and this is valid for coefficients in any principal ideal domain. The description of the boundary map I give however will work only when $\mathbb{Z}_2$ coefficients are considered.

In the first half of the paper the main goal will be to establish a version of the **Morse Homology Theorem**, following the approach of Abbondandolo and Majer [AM]. We begin in a general setting: working on a Hilbert manifold $\mathcal{L}$, we show that a Morse vector field $X$ satisfying certain conditions, together with a Morse-Smale metric $\mathcal{G}$ determine a chain complex $CM_*(\mathcal{L}, X, \mathcal{G})$, where $CM_k(\mathcal{L}, M, \mathcal{G})$ is the free abelian group generated by the rest points of $X$ of **Morse index** $k$. Next, we show that the homology of this complex is isomorphic to the singular homology $H_*^{\text{sing}}(\mathcal{L})$.

Having done this, we turn to the situation at hand. In this essay we will be interested in two particular types of time dependent **Hamiltonians** $H : T^*M \to \mathbb{R}$. The first is of the basic form kinetic plus potential energy. More specifically, let $V : S^1 \times M \to \mathbb{R}$ denote a smooth time dependent potential and define

$$H : T^*M \to \mathbb{R}, \qquad H(t, x, p) = \frac{1}{2}|p|^2 - V(t, x), \tag{1.1}$$

where the norm $|\cdot|$ is the norm induced on $T^*M$ by the Riemannian metric $g$ on $M$.

Now consider the **Lagrangian** $L : TM \to \mathbb{R}$ obtained from $H$ by the **Legendre transformation**. Given the form we have insisted $H$ takes, the Legendre transformation is particularly simple, and $L$ is of the form kinetic minus potential energy, that is,

$$L : TM \to \mathbb{R}, \qquad L(t, x, v) = \frac{1}{2}|v|^2 - V(t, x),$$

$L$ determines an **action functional** $\mathcal{S}$ defined on the free loop space $\mathcal{L}M$ of $M$, defined by

$$\mathcal{S}(x) = \int_0^1 L\left(t, x(t), \dot{x}(t)\right) dt = \int_0^1 \left(\frac{1}{2}|\dot{x}(t)|^2 - V(t, x(t))\right) dt, \qquad \text{for } x : S^1 \to M.$$

Letting $X = -\nabla\mathcal{S}$, the negative gradient flow of $\mathcal{S}$, we show that $X$ satisfies the conditions required above, and we conclude that the chain complex, which we shall denote by $CM_*\left(\mathcal{L}M, L, \mathcal{G}\right)$, is isomorphic to the singular homology of the free loop space (and thus independent of the choice of $\mathcal{S}$).

$H$ also determines an action functional $\mathcal{A}$ on the free loop space $\mathcal{L}T^*M$ defined by

$$\mathcal{A}(a) = \int_{S^1} a^*(\theta) - \int_0^1 H(t, a(t)), \qquad \text{for } a : S^1 \to T^*M,$$

where $\theta$ is the **Liouville** 1**-form** on $T^*M$.

It is tempting to try and proceed as before and set up the Morse homology of $\mathcal{A}$ on $T^*M$. Unfortunately this fails; in contrast to $\mathcal{S}$ the action functional $\mathcal{A}$ is in general not bounded below, nor do its critical points admit finite Morse indices. Floer however got round this problem by instead viewing the negative gradient equation as an elliptic PDE for maps from a cylinder to $T^*M$. This allowed him to set up a chain complex generated by the critical points of $\mathcal{A}$, graded by their **Conley-Zehnder index**. Floer however worked with a closed symplectic manifold; since $T^*M$ is not compact more work is required. In particular proving compactness of the moduli spaces is much harder. We outline the approach of Abbondandolo and Schwarz [AS] that allows us to get round these difficulties. This allows us to set up the **Floer homology** of $H$ on $T^*M$.

We will also study time dependent **twisted magnetic Hamiltonians** $H^{\mathrm{tw}} : T^*M \to \mathbb{R}$. Here we let $\alpha$ denote a 1-form on $M$ and modify our definition of $H$ to become

$$H^{\mathrm{tw}} : T^*M \to \mathbb{R}, \qquad H^{\mathrm{tw}}(t, x, p) = \frac{1}{2}\left|p + \alpha_x\right|^2 + V(t, x). \tag{1.2}$$

We quickly motivate this choice of Hamiltonian. If $\omega$ denotes the canonical symplectic form $d\theta = dp \wedge dx$, then given a closed 2-form $\sigma$ on $M$ we can consider the **twisted symplectic form** $\Omega := \omega + \tau^*\sigma$, where $\tau : T^*M \to M$ is the footpoint map. We will show that the Floer homology groups $HF_*(T^*M, \Omega, H, J)$ coincide with the groups $HF_*(T^*M, \omega, H^{\mathrm{tw}}, J)$, and thus knowlege of the latter allows us to also study the Floer homology of $T^*M$ with respect to the twisted symplectic structure.

One of the great strengths of Floer homology is that it is in fact independent of the choice of $H$; this is **Floer's Continuation Principle**. We won't actually need this result, as we shall deduce the independence indirectly. Indeed, the final part of the paper is devoted to proving the existence of a chain isomorphism, due to Abbondandolo and Schwarz, between the Morse complex $CM_*\left(\mathcal{L}M, L, \mathcal{G}\right)$ and the Floer complex $CF_*\left(T^*M, \omega, H, J\right)$. This allows us to deduce the result, originally proved by Viterbo [V], and also by Salamon and Weber [SW] that the Floer homology of the cotangent bundle of $M$ under the canonical symplectic structure is isomorphic to the singular homology of the free loop space (and thus in particular is independent of the choice of $H$).

**Notation**

Points in $M$ will be denoted by $x, y$ etc. Points in $T^*M$ will be denoted by $(x, p)$ with $p \in T_x^*M$. Similarly points in $TM$ will be denoted by $(x, v)$ with $v \in T_xM$. Loops on $M$ will be denoted by $x(t), y(t)$ etc., and loops on $T^*M$ will be denoted by $a(t), b(t)$ etc., where $a(t) = (x(t), p(t))$ (using local triviality of the cotangent bundle), so $p(t) \in T_{x(t)}^*M$ is a covector field along $x(t)$.

The notation '$\doteqdot$' is explained in (3.2) at the start of §3.

**All homology groups have coefficients in $\mathbb{Z}_2$.**

*Acknowlegements.* My main reference in writing this paper has obviously been [AS]. The expository article [W3] was very informative and helped me get started. In fact, I particularly recommend this article as it summarizes the three proofs currently known for this result. In §2 I followed the approach of [AM], §2, although the presentation there was considerably more general than I needed, and hence I simplified

it somewhat. The proof of Theorem 2.9 is based on [Lau], Theorem 3.5. In §3 I found Weber's thesis [W1], Appendix *B* very helpful; the proof of Proposition 3.4 is taken from there. The proof of Theorem 3.12 comes from [W2], Theorem 2.2. The treatment in §4 was strongly influenced by the lecture notes by Salamon [Sa]. I used the excellent [Sc3], §3 and [Sc2], §2 for the treatment of Theorem 4.7 and Theorem 4.23; the former offers an extremely lucid exposition, and the latter gives detailed proofs. The discussion I give of the compactified moduli spaces is based on that in [Ke]. §5 is entirely based on §3 of [AS]. I make no claim of originality; several of the proofs in §3 are my own, but there is nothing new here. Finally I would like to thank Dr. Paternain for suggesting I tackle twisted symplectic structures, and more generally, for proposing this essay and thus introducing me to this fascinating area of mathematics.

## 2. Morse Homology

**Definitions**

We will assume that the reader is familiar with the concept of Hilbert and Banach manifolds; good references for this are [K] and [Lan].

Suppose $\mathcal{L}$ is a Hilbert manifold and $X : \mathcal{L} \to T\mathcal{L}$ a vector field on $\mathcal{L}$. We quickly recap what it means to say that $X$ is a Morse vector field. Let $\varphi^t$ denote the flow of $X$, and let $\mathcal{E}(X)$ denote the domain of $X$, that is, the points $(t, x)$ in $\mathbb{R} \times \mathcal{L}$ for which $\varphi^t(x)$ is defined. More explicitly, let $\tau^+, \tau^- : M \to \bar{\mathbb{R}}$ (where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$) denote the functions such that

$$\mathcal{E}(X) = \{(t, x) \mid \tau^-(x) \le t \le \tau^+(x), x \in \mathcal{L}\}.$$

$\mathcal{E}(X)$ is an open neighborhood of $\{0\} \times \mathcal{L}$ in $\mathbb{R} \times \mathcal{L}$, and the functions $\tau^+, \tau^-$ are upper and lower semi-continuous respectively; see [Lan], Chapter IV, §2. $X$ is called **complete** if $\mathcal{E}(X) = \mathbb{R} \times \mathcal{L}$.

A **rest point** of $X$ is a point $x \in \mathcal{L}$ such that $X(x) = 0$. Let $\mathrm{rest}(X)$ denote the set of rest points of $X$. If $x$ is a rest point of $X$ then $x$ is fixed point of $\varphi^t$ for any $\tau^-(x) \le t \le \tau^+(x)$. We say $x$ is a **hyperbolic** rest point of $X$ if the linear operator $DX_x : T_x\mathcal{L} \to T_x\mathcal{L}$ has its spectrum disjoint from the real axis (where we identify $T_{X(x)}T_x\mathcal{L}$ with $T_x\mathcal{L}$). $X$ is a called a **Morse** vector field if all its rest points are hyperbolic. Note that this implies that the rest points are isolated.

If $x \in \mathcal{L}$ is a hyperbolic rest point of $X$, then the Spectral Decomposition Theorem gives a splitting of the Hilbert space $H_x = T_x\mathcal{L}$ as $H_x = H_x^u \oplus H_x^s$ corresponding to the partition of the spectrum of $T := DX_x$ into the closed subsets $\sigma^u(T) := \sigma(T) \cap \{\mathrm{Re}(z) > 0\}$ and $\sigma^s(T) := \{\sigma(T) \cap \{\mathrm{Re}(z) < 0\}$. $H_x^u$ and $H_x^s$ are closed $D\varphi^t$-invariant linear subspaces for any $\tau^-(x) \le t \le \tau^+(x)$. We will frequently identify $H_x$ with $H_x^u \times H_x^s$. The **Morse index** of the point $x$, $m(x)$ is the dimension (which is not necessarily finite) of $H_x^u$.

A **Lyapunov** function $f$ for $X$ is a $C^1$ function $f : \mathcal{L} \to \mathbb{R}$ that is strictly decreasing along flow lines of $X$. More precisely we require that for any $y \in \mathcal{L} \backslash \mathrm{rest}(X)$, $df_y(X(y)) < 0$. Note that this implies that the set of critical points of $f$, $\mathrm{crit}(f) := \{x \in \mathcal{L} \mid df_x = 0\}$ is contained in $\mathrm{rest}(X)$. In fact, if $X$ is Morse, then we have equality (see [AM], p56).

Suppose $x \in \mathcal{L}$ is a hyperbolic rest point of $X$. We define the **unstable** and the **stable manifolds** of the rest point $x$ to be the subsets of $\mathcal{L}$,

$$W^u(x) := \left\{ y \in \mathcal{L} \,\middle|\, \tau^-(y) = -\infty, \lim_{t \to -\infty} \varphi^t(y) = x \right\}.$$

$$W^s(x) := \left\{ y \in \mathcal{L} \,\middle|\, \tau^+(y) = \infty, \lim_{t \to \infty} \varphi^t(y) = x \right\}.$$

The crucial result we need from the theory of hyperbolic dynamical systems is the Global Unstable Manifold Theorem. Proofs can be found in (for example) [AM], Theorem 1.20 or [BH], Theorem 4.2.

**Theorem 2.1. (The Global Unstable Manifold Theorem)**

*Let $\mathcal{L}$ be a Hilbert manifold and $X$ a vector field on $\mathcal{L}$. Let $x$ be a hyperbolic rest point of $X$ and let $H_x$ denote the Hilbert space $T_x\mathcal{L}$. Then the stable and unstable manifolds $W^u(x)$ and $W^s(x)$ are the images of injective immersions*

$$e^u : H_x^u \to \mathcal{L}, \quad e^s : H_x^s \to \mathcal{L},$$

*such that $e^u(0) = e^s(0) = 0$ and $De_0^u = De_0^s = \mathbb{1}$. Thus $W^u(x)$ and $W^s(x)$ are immersed submanifolds of $\mathcal{L}$ with*

$$\dim W^u(x) = \mathrm{codim} W^s(x) = m(x).$$

*Moreover, if $X$ admits a Lyapunov function and $x$ has finite Morse index then the maps $e^s, e^u$ are embeddings and thus $W^u(x)$ and $W^s(x)$ are embedded submanifolds of $\mathcal{L}$.*

### Broken orbits and filtration functions

We will now prove various results on certain special types of Morse vector fields that will allow us to construct the Morse complex in the next subsection.

**Definition 2.2.** The **Palais-Smale** condition: this is a condition on the Lyapunov function $f$. A Palais-Smale sequence $(x_n) \in \mathcal{L}$ is a sequence such that $f(x_n)$ is convergent and $df_{x_n}(X(x_n)) \to 0$. We say that $f$ satisfies the Palais-Smale condition if every Palais-Smale sequence contains a convergent subsequence.

For the remainder of this subsection we shall assume the following:

**Assumption 2.3.** *$X$ is a complete Morse vector field on $\mathcal{L}$ such that all of its rest points have finite Morse index, and such that $X$ admits a Lyapunov function $f$ that is bounded below and satisfies the Palais-Smale condition.*

**Lemma 2.4.** *For any $\alpha \leq \beta \in \mathbb{R}$ the set $\mathrm{rest}(X) \cap f^{-1}([\alpha,\beta])$ is finite.*

*Proof.* It suffices to show that $\mathrm{rest}(X) \cap f^{-1}([\alpha,\beta])$ is compact, since the rest points are isolated. Suppose $(x_n) \in \mathrm{rest}(X) \cap f^{-1}([\alpha,\beta])$. Then by compactness $(f(x_n))$ has a convergent subsequence $(f(x_{n(k)}))$, and since the $x_{n(k)}$ are rest points, $df_{x_{n(k)}}(X(x_{n(k)})) = 0$, and hence $(x_{n(k)})$ is a Palais-Smale sequence. The result follows. ■

**Lemma 2.5.** *Suppose $\alpha \leq \beta \in \mathbb{R}$, $(x_n) \in \mathcal{L}$ and $(t_n) \geq 0$. Set*

$$I_n := \left\{ \varphi^t(x_n) \;\middle|\; 0 \leq t \leq t_n \right\}, \qquad J_n := \left\{ \varphi^t(x_n) \;\middle|\; -t_n \leq t \leq 0 \right\}.$$

*Suppose that $\alpha \leq f(I_n) \leq \beta$ for each $n$, and that*

$$\lim_{n\to\infty} \frac{f(x_n) - f(\varphi^{t_n}(x_n))}{t_n} = 0. \tag{2.1}$$

*Then there exists an increasing subsequence $n(i)$ and points $y_{n(i)} \in I_{n(i)}$ such that $y_{n(i)}$ is convergent.*

*Similarly if $\alpha \leq f(J_n) \leq \beta$ for each $n$ and (2.1) holds then there exists an increasing subsequence $n(j)$ and points $z_{n(j)} \in J_{n(j)}$ such that $z_{n(j)}$ is convergent*

*Proof.* By the Mean Value Theorem, there exists $t_n' \in (0, t_n)$ such that

$$df\left(X\left(\varphi^{t_n'}(x_n)\right)\right) = \frac{f(x_n) - f(\varphi^{t_n}(x_n))}{t_n}.$$

Setting $y_n := \varphi^{t_n'}(x_n)$, we see that $(y_n)$ is a Palais-Smale sequence, and the claim follows. The proof of the second assertion is similar. ■

**Proposition 2.6.** *Let $x \in \mathcal{L}$. Then there exists $y, z \in \mathrm{rest}(X)$ such that $\varphi^t(x) \to y$ as $t \to \infty$ and $\varphi^t(x) \to z$ as $t \to -\infty$.*

*Proof.* We may of course assume that $x \notin \mathrm{rest}(X)$. Set $\alpha := \inf f > -\infty$ and $\beta := f(x)$. Let $x_n = x$ for $n \in \mathbb{N}$. By Lemma 2.5, there exists $(t_n) \geq 0$ such that $(\varphi^{t_n}(x))$ is a Palais-Smale sequence. Since $df_x \neq 0$ but $df_x(X(\varphi^{t_n}(x))) \to 0$, we must have $X(\varphi^{t_n}(x)) \to 0$. The Palais-Smale condition then implies the existence of an increasing sequence $n(k)$ and $y \in \mathcal{L}$ such that $\varphi^{t_{n(k)}}(x) \to y$, and then since $X$ is smooth, $X(y) = \lim_{k\to\infty} X(\varphi^{t_{n(k)}}(x)) = 0$, and so $y \in \mathrm{rest}(X)$. If $(t_n)$ is bounded then passing to a subsequence if necessary we may assume that $t_n \to s$ for some $s \geq 0$. But then $\varphi^s(x) = y$, and thus $\varphi^{t+s}(x) = \varphi^t(y) = y$ and the result is clear. Thus we may assume that $t_n \to \infty$.

Select $r > 0$ such that

$$\overline{B_{2r}(y)} \cap \mathrm{rest}(X) = \{y\} \quad \text{and} \quad \sup_{z\in B_{2r}(y)} |X(z)| \leq 1$$

(where the ball is taken with respect to the geodesic metric on $\mathcal{L}$). If $\varphi^t(x)$ does not converge to $y$ as $t \to \infty$, then since $t \mapsto \varphi^t(x)$ is continuous and $\varphi^{t_{n(k)}}(x) \to y$ as $k \to \infty$, we may choose sequences $(s_m), (s_m') \geq 0$ such that:

(i) $s_m \leq s_m' \leq s_{m+1}$, $s_m \to \infty$,
(ii) $\varphi^{s_m}(y) \in \partial B_{2r}(y)$ and $\varphi^{s_m'}(x) \in \partial B_r(y)$,
(iii) for $s_m \leq t \leq s_m'$, $\varphi^t(x) \in \overline{B_{2r}(y) \backslash B_r(y)}$,

(iv) if $r_m := s'_m - s_m$ then there exists $\epsilon > 0$ such that for all $m \in \mathbb{N}$, $r_m > \epsilon$.

Then observe that since $t \mapsto f(\varphi^t(x))$ is continuous,

$$\lim_{m\to\infty} f(\varphi^{s_m}(x)) = \lim_{m\to\infty} f\left(\varphi^{s'_m}(x)\right) = \lim_{k\to\infty} f\left(\varphi^{t_{n(k)}}(x)\right) = f(y),$$

and thus

$$\lim_{m\to\infty} \frac{f\left(\varphi^{s'_m}(x)\right) - f(\varphi^{s_m}(x))}{r_m} = 0.$$

Then Lemma 2.5 implies that we can find a a Palais-Smale sequence $(z_{m(k)})$ converging to a rest point $z \neq y \in \overline{B_{2r}(y)\backslash B_r(y)}$. This contradiction proves that $\varphi^t(x) \to y$ as $t \to \infty$. The proof that there exists $z \in \text{rest}(X)$ such that $\varphi^t(x) \to z$ as $t \to -\infty$ is similar. ∎

The proofs of the next two results are similar, and are omitted. See [AM], Proposition 2.2(ii) and Corollary 2.4(i) for more details. Observe that the first result pertains to the forward orbit only.

**Proposition 2.7.** *Let $x_n \to x \in \mathcal{L}$ and $(t_n) \geq 0$ be any sequence. Then there exists an increasing subsequence $n(k)$ and $y \in \mathcal{L}$ such that $\varphi^{t_{n(k)}}(x_{n(k)}) \to y$.*

In the following, given $x, y \in \text{rest}(X)$, let $\mathcal{W}(x,y) := W^u(x) \cap W^s(y)$.

**Proposition 2.8.** *Let $x_n \to x \in \mathcal{L}$. Then there exists an increasing sequence $n(k)$ and $y, z \in \text{rest}(X)$ such that $x_{n(k)} \in \mathcal{W}(y,z)$ for all $k$.*

We shall meet precompactness results several times throughout this paper. Here is the first.

**Corollary 2.9. (Precompactness)**
*Let $x \in \text{rest}(X)$. Then the unstable manifold $W^u(x)$ is precompact in $\mathcal{L}$.*

*Proof.* Let $(y_n) \in W^u(x)$. Set $z_n := \varphi^{-n}(y_n)$ and $t_n := n$. Then $\varphi^{t_n}(z_n) = y_n$, and Proposition 2.7 gives the existence of an increasing sequence $n(k)$ such and $y \in \mathcal{L}$ such that $y_{n(k)} \to y$. Then $y \in \overline{W^u(x)}$ as the latter is closed. Since $W^u(x)$ is finite dimensional by the Global Unstable Manifold Theorem 2.1, this is enough to conclude that $\overline{W^u(x)}$ is sequentially compact. ∎

Note that $\mathcal{W}(x,y)$ is also precompact, as $\overline{\mathcal{W}(x,y)}$ is a closed subset of a compact space. The following result is the main result of this section, and like Corollary 2.9 we shall see this in various guises throughout the paper. The statement of the result is rather complicated; essentially this is saying that if $x_n \to x \in \text{rest}(X)$ is any sequence[1], then, passing to a subsequence if necessary, the forward orbit under $\varphi$ of the $x_n$ converges to a 'broken orbit' consisting of $\ell + 1$ flow lines.

**Theorem 2.10. (Broken Orbit Lemma)**
*Let $x_n \to x \in \text{rest}(X)$ be a nonconstant sequence. Then there exists:*

(1) *an increasing subsequence $n(k)$,*
(2) $y = y_0, y_1, \ldots, y_\ell, y_{\ell+1} = z \in \text{rest}(X)$ *with* $f(z) < f(y_\ell) < \cdots < f(y_1) < f(y)$,
(3) *sequences $(t_n^i)$ for $i = 1, \ldots, \ell$ such that $t_n^i > t_n^{i-1}$ for all $k \in \mathbb{N}$,*
(4) $w_1, \ldots, w_{\ell+1} \in \mathcal{L}$ *such that* $w_i \in \mathcal{W}(y_{i-1}, y_i)$ *for* $i = 1, \ldots, \ell+1$,

*such that:*

(a) $x_{n(k)} \in \mathcal{W}(y,z)$ *for all* $n(k)$,
(b) $\lim_{k\to\infty} \varphi^{t^i_{n(k)}}(x_{n(k)}) = w_i$ *for* $i = 1, \ldots, \ell$,
(c) $x = y_k$ *for some* $0, \ldots, \ell+1$.

To reduce the number of superscripts, for the purposes of this proof we will write $\varphi(t,x)$ instead of $\varphi^t(x)$.

*Proof.* By Proposition 2.8 we may assume (up to passing to a subsequence) that $x_n \in \mathcal{W}(y,z)$ for some $y, z \in \text{rest}(X)$. Choose a regular value $c_1$ of $f$ such that there are no critical points of $f$ in the strip $[c_1, f(y))$. Now choose $(t_n^1) \geq 0$ such that $f(\varphi(t_n^1, x_n)) = c_1$. By Proposition 2.7 we can find $n_1(k)$ and $w_1 \in \mathcal{L}$ such that

$$\varphi\left(t^1_{n_1(k)}, x_{n_1(k)}\right) \to w_1.$$

Let $y_0, y_1 \in \text{rest}(X)$ be such that $w_1 \in \mathcal{W}(y_0, y_1)$. Then actually $y_0 = y$; the argument for this is similar to the (forthcoming) statement that $y'_1 = y_1$ and is omitted. Observe that the following statement holds:

[1]The hypothesis that the $(x_n)$ tend towards a rest point is not necessary for a result of this type; it just simplifies the statement. Since the statement is already complicated, and we only need the case where $x$ is a rest point, we are content to stick to this special case.

$$f(c_1) = \inf_{\rho\geq 0}\lim_{n\to\infty} f\left(\varphi\left(t_n^1+\rho, x_n\right)\right). \tag{2.2}$$

Now choose a regular value $c_2$ of $f$ such that there are no critical points of $f$ in the strip $[c_2, f(y_1))$. Choose a subsequence $(t_n^2) \geq 0$ such that $f(\varphi(t_n^2, x_n)) = c_2$. Then as before we may choose a subsequence $n_2(k)$ of $n_1(k)$ and $w_2 \in \mathcal{L}$ such that

$$\varphi\left(t^2_{n_2(k)}, x_{n_2(k)}\right) \to w_2.$$

Now suppose $w_2 \in \mathcal{W}(y_1', y_2)$. We will show that $y_1' = y_1$; this is the crux of the proof. First we claim that $f(y_1') = f(y_1)$.

Suppose that $r_n := t_n^2 - t_n^1$ is bounded. Then upon passing to a subsequence, $r_n \to r$, and then restricting to an appropriate subsequence,

$$c_2 = f\left(\varphi\left(t_n^1 + r_n, x_n\right)\right) \to f(\varphi(r, w_1) > f(y_1),$$

contradicting $c_2 < f(y_1)$. Hence for every $\rho > 0$ and $n$ big enough we have

$$f\left(\varphi\left(t_n^2 - \rho, x_n\right)\right) < f\left(\varphi\left(t_n^1 + \rho\right)\right),$$

and (2.2) implies that $f(y_1') \leq f(y_1)$. Since $[c_2, f(y_1))$ contains no critical values, we must have $f(y_1') = f(y_1)$.

Now we show $y_1' = y_1$. Suppose not. Since rest points are isolated, we can choose disjoint open balls $B, B'$ about $y_1$ and $y_1'$ respectively whose closures contain no other critical points of $f$. There exists $s_0 \geq 0$ such that for every $s \geq s_0$,

$$\varphi(s, w_1) \in B, \qquad \varphi(-s, w_2) \in B'.$$

Thus for $k$ large enough,

$$\varphi(t^1_{n_2(k)} + s_0, x_{n_2(k)}) \in B, \qquad \varphi\left(t^2_{n_2(k)} - s_0, x_{n_2(k)}\right) \in B'.$$

Hence for all $k$ large, we can find $s_k \in (t^1_{n_2(k)} + s_0, t^2_{n_2(k)} - s_0)$ such that $\varphi(s_k, x_{n_2(k)}) \in \partial B$. Then by Proposition 2.7 again we can on passing to a further subsequence (which I shall still denote by $n_2(k)$) assume there exists $w' \in \partial B$ such that $\varphi(s_k, x_{n_2(k)}) \to w'$. Now the sequence $\gamma_k := s_k - t^1_{n_2(k)}$ is unbounded, lest passing to a subsequence we have $\gamma_k \to \gamma$, and this contradicts the fact that $\varphi(s, w_1) \notin \partial B$ for all $s \geq s_0$. Similarly $t^2_{n_{2(k)}} - s_k$ is unbounded, and we discover

$$f(y_1) \geq \sup_{t\in\mathbb{R}} f(\varphi(t, w')) \geq \inf_{t\in\mathbb{R}} f(\varphi(t, w')) = f(y_1') = f(y_1).$$

Thus $\varphi(\cdot, w')$ is constant and so $w'$ is a rest point of $X$ lying in $\partial B$; contradiction.

The proof is basically now complete. Using the fact that $f$ is bounded below, by repeating this argument we find rest points $y = y_0, y_1, \dots, y_\ell, y_{\ell+1} = z$ and subsequences $(t_n^i)$ and points $w_i \in \mathcal{W}(y_{i-1}, y_i)$ as in the statement of the theorem, with a few changes in the last case.

Now we certainly have

$$f(z) \leq f(x) \leq f(y),$$

and thus since $x$ is a rest point, $f(x) = f(y_k)$ for some $k$, whence by an argument similar to that above, $x = y_k$ for some $k$. ∎

An important consquence of this result is the fact that $\mathcal{W}(y_i, y_{i-1}) \neq \emptyset$; the Morse-Smale condition below will show why this is important.

Let us now introduce another condition we like to impose on $X$.

**Definition 2.11.** The **Morse-Smale** condition is that for all $x, y \in \mathrm{rest}(X)$ such that $m(x) \leq m(y) + 1$, the unstable manifold $W^u(x)$ and the stable manifold $W^s(y)$ are transverse.

For the remainder of this subsection let us assume the Morse-Smale condition in addition to Assumption 2.3. This allows us to deduce the next result, which is another result we shall see again in a different guise when we come to construct the Floer complex in §4.

**Proposition 2.12.** *Let $x, y \in \mathrm{rest}(X)$. If $m(x) \leq m(y)$, then*

$$\mathcal{W}(x, y) = \begin{cases} \emptyset & x \neq y \\ \{x\} & x = y. \end{cases}$$

*If $m(x) = m(y) + 1$ then $\mathcal{W}(x, y)$ consists of finitely many (possibly zero) flow lines.*

*Proof.* Firstly $W^u(x) \cap W^s(x) = \{x\}$, since $f$ decreases along gradient lines (and note such an intersection is always transverse). If $x \neq y$ then $W^u(x) \pitchfork W^s(y)$ implies that if $\mathcal{W}(x, y) \neq \emptyset$ then $\mathcal{W}(x, y)$ is a submanifold of $\mathcal{L}$ and

$$\dim(\mathcal{W}(x, y)) = \dim(T^u_x\mathcal{L}) - \operatorname{codim}(T^s_x\mathcal{L}) = m(x) - m(y) \leq 0.$$

Hence $\mathcal{W}(x, y) \neq \emptyset$ implies that $m(x) = m(y)$ and thus $\mathcal{W}(x, y)$ has dimension 0. But now if $z \in \mathcal{W}(x, y)$ then $\varphi(t, z) \in \mathcal{W}(x, y)$ for all $t \in \mathbb{R}$ by definition of $\mathcal{W}(x, y)$, and thus $\dim(\mathcal{W}(x, y)) \geq 1$. Contradiction. This proves the first claim.

To prove the second claim, in this case the argument above shows that if $\mathcal{W}(x, y) \neq \emptyset$ then $\dim(\mathcal{W}(x, y)) = 1$ and thus $\mathcal{W}(x, y)$ is a discrete set of flow lines. Observe that $\mathcal{W}(x, y) \cup \{x, y\}$ is closed, since if $(z_n) \in \mathcal{W}(x, y) \cup \{x, y\}$ tended to some $z \notin \mathcal{W}(x, y) \cup \{x, y\}$ then $z$ would lie on a flow line that tended to some $w \neq x, y$, and then the Broken Orbit Lemma would imply $m(y) < m(w) < m(x)$, contradiction. Thus $\overline{\mathcal{W}(x, y)} = \mathcal{W}(x, y) \cup \{x, y\}$ is compact by Corollary 2.9, and hence $\mathcal{W}(x, y)$ must contain only finitely many flow lines. ■

If $A \subseteq \mathcal{L}$, denote by $\varphi^+(A)$ the subset

$$\varphi^+(A) = \left\{ y \in \mathcal{L} \mid y = \varphi^t(x) \text{ for some } x \in A \right\}.$$

**Corollary 2.13.** *Let $x, y \in rest(X)$, $x \neq y$ with $m(x) \leq m(y)$. Then there exists $r > 0$ such that*

$$\varphi^+(B_r(x)) \cap B_r(y) = \emptyset.$$

*Proof.* If not, then there exists $(x_n) \in \mathcal{L}$ and $(t_n) \geq 0$ such that $x_n \to x$ and $\varphi^{t_n}(x_n) \to x$. By the Broken Orbit Lemma we can find rest points $y = y_0, \ldots, y_k = x$ with $k \geq 2$ and $\mathcal{W}(y_i, y_{i-1}) \neq \emptyset$ for $i = 1, \ldots, k-1$. Proposition 2.12 then implies that $m(y) < m(y_1) < \cdots < m(x)$, a contradiction. ■

We will now impose one more condition. This condition is not necessary, and at the end of this section we will indicate how to remove it.

Let us suppose that there are **only finitely many rest points of any given index**, that is, for any $j \in \mathbb{N}$ there are at most finitely many $x \in \operatorname{rest}(X)$ such that $m(x) = j$.

This extra assumption allows us to deduce:

**Corollary 2.14.** *If $y \in \operatorname{rest}(X)$,*

$$y \notin \overline{\bigcup_{\substack{x \neq y \in \operatorname{rest}(X) \\ m(x) \leq m(y)}} W^u(x)}.$$

*Proof.* Immediate from Corollary 2.9, since the union on the right-hand side is a union of finitely many sets, and hence is equal to $\bigcup_{\substack{x \neq y \in \operatorname{rest}(X) \\ m(x) \leq m(y)}} \overline{W^u(x)}$. ■

The main reason for wanting to impose that there only exist finitely many rest points of any given index is that this allows us to deduce the existence of a special function associated to $X$ that we will shortly use to create the Morse complex of $X$.

**Theorem 2.15. (Existence of filtration function)**
*There exists a function $F : \operatorname{rest}(X) \to (0, \infty)$ such that for any rest points $x, y$ such that $x \neq y$ and $m(x) \leq m(y)$ we have*

$$\varphi^+(B_{F(x)}(x)) \cap B_{F(y)}(y) = \emptyset.$$

We call $F$ a **filtration function** for $X$. The following proof is sadly nonconstructive; it proves existence only.

*Proof.* The previous corollary gives a function $F_1 : \operatorname{rest}(X) \to (0, \infty)$ such that

$$\bigcup_{\substack{x \neq y \in \operatorname{rest}(X) \\ m(x) \leq m(y)}} W^u(x) \cap B_{F_1(y)}(y) = \emptyset.$$

We will show that there exists a function $F_2 : \operatorname{rest}(X) \to (0, \infty)$ such that if $x, y$ are distinct rest points with $m(x) \leq m(y)$ then

$$\varphi^+(B_{F_2(x)}(x)) \cap B_{F_1(y)}(y) = \emptyset.$$

The desired function $F$ is then defined by $F(x) := \min\{F_1(x), F_2(x)\}$.

Suppose no such function $F_2$ exists. Then there exists $x, y \in \operatorname{rest}(X)$ with $m(x) \leq m(y)$, and $(x_n) \in \mathcal{L}$, $(t_n) \geq 0$ such that $x_n \to x$, and $\varphi^{t_n}(x_n) \in B_{F_1(y)}(y)$ for all $n$. But then by the Broken Orbit Lemma we can find rest points $z \neq z_1, \ldots, z_k = x$ with $\mathcal{W}(z_i, z_{i-1}) \neq \emptyset$ and $W^u(z_1) \cap B_{F_1(y)}(y) \neq \emptyset$. Then Proposition 2.12 implies that $m(z_1) \leq m(x) \leq m(y)$, and since $z_1 \neq y$, this contradicts the definition of $F_1$. ■

**The Morse complex**

We conclude this section by constructing a chain complex associated to $X$ and a filtration function $F$ for $X$, and then showing that its homology is isomorphic to the singular homology of $\mathcal{L}$. We will complete the story by stating a result that shows that a choice of orientation of each unstable manifold $W^u(x)$ for $x \in \text{rest}(X)$ gives an isomorphism between this chain complex and a chain complex with chain groups the free abelian groups generated by the rest points of $X$.

As before, in addition to Assumption 2.3, the Morse-Smale condition and that there are only finitely many rest points of a given index. Let $F : \text{rest}(X) \to (0, \infty)$ be a filtration function for $X$. Set $\text{rest}_k(X) := \{x \in \text{rest}(X) | m(x) = k\}$ and

$$L^k := \bigcup_{x \in \text{rest}_\ell(X), \ell \leq k} \varphi^+ \left(\text{int} B_{F(x)}(x)\right)$$

for $k \geq 0$, and $L^k := \emptyset$ for $k < 0$. Set $L^\infty := \bigcup_{k \in \mathbb{Z}} L^k$. The first result is the following.

**Proposition 2.16.** *The inclusion $L^\infty \hookrightarrow \mathcal{L}$ is a homotopy equivalence.*

*Proof.* Proposition 2.6 implies that for all $x \in \mathcal{L}$,

$$\inf \left\{ t \in [0, \infty) \mid \varphi^t(x) \in L^\infty \right\} < \infty$$

Select a continuous function $h : \mathcal{L} \to [0, \infty)$ such that

$$\inf \left\{ t \in [0, \tau^+(x)) \mid \varphi^t(x) \in L^\infty \right\} < h(x), \qquad \text{for all } x \in \mathcal{L}.$$

Now define a map $g : \mathcal{L} \to L^\infty$ by $g(x) = \varphi^{h(x)}(x)$, and claim that $g$ is a homotopy inverse to the inclusion $i : L^\infty \hookrightarrow \mathcal{L}$. Indeed, if $G : \mathcal{L} \times I \to \mathcal{L}$ is the map $(x, s) \mapsto \varphi^{sh(x)}(x)$ then $G(x, 0) = \mathbb{1}_{\mathcal{L}}$ and $G(x, 1) = i \circ g$, and, noting that $G$ restricts to a map $L^\infty \times I \to I$, the restriction of $G$ gives a homotopy from $i \circ g$ to $\mathbb{1}_{L^\infty}$. ■

The second crucial result is that $\left\{L^k\right\}_{k \in \mathbb{Z}}$ is a **cellular filtration** of $L^\infty$.

**Proposition 2.17.** *$\left\{L^k\right\}_{k \in \mathbb{Z}}$ is a cellular filtration of $L^\infty$. That is,*

$$C_*^{\text{sing}}(L^\infty) = \bigcup_{k \in \mathbb{Z}} C_*^{\text{sing}}(L^k)$$

*(where $C_*^{\text{sing}}(L^\infty)$ denotes the singular chain complex of $L^\infty$) and*

$$H_i^{\text{sing}}\left(L^k, L^{k-1}\right) = 0 \textit{ for } i \neq k.$$

This is a non-trivial and we shall only indicate what needs to be done to prove the result. The rest of the details can be found in [AM], Theorem 2.8(ii).

*Proof. (sketch)*

It is clear that $\{L^k\}_{k \in \mathbb{Z}}$ is an open covering of $L^\infty$, and thus it is enough to verify that $H_i^{\text{sing}}(L^k, L^{k-1}) = 0$ if $i \neq k$. Let $U^k$ denote the open set

$$U^k := \bigcup_{x \in \text{rest}_k(X)} \varphi^+ \left(\text{int} B_{F(x)}(x)\right).$$

Since $L^k = L^{k-1} \cup U^k$, by excision we have $H_i^{\text{sing}}(L^k, L^{k-1}) \cong H_i^{\text{sing}}(U^k, U^k \cap L^{k-1})$. Now if

$$U(x) := \varphi^+ \left(\text{int} B_{F(x)}(x)\right),$$

then since $F$ is a filtration function, the $U(x)$ are pairwise disjoint for $x \in \text{rest}_k(X)$ and

$$U^k = \bigoplus_{x \in \text{rest}_k(X)} U(x).$$

Thus

$$H_i^{\text{sing}}\left(U^k, U^k \cap L^{k-1}\right) \cong \bigoplus_{x \in \text{rest}_k(X)} H_i^{\text{sing}}\left(U(x), U(x) \cap L^{k-1}\right).$$

The proof is then completed by showing that $(U(x), U(x) \cap L^{k-1})$ is homotopy equivalent to $(D^k, S^{k-1})$, as then we certainly have $H_i^{\text{sing}}(P^k, P^{k-1})$ non-zero only if $i = k$ (in fact, this tells us far more, as we shall see in the Morse Homology Theorem below) . ■

By the **Cellular Filtration Theorem** (see [R], Theorem 8.36), it follows that if $C_*^{\text{cell}}(L^\infty)$ denotes the cellular chain complex associated to the filtration $\left\{L^k\right\}_{k\in\mathbb{Z}}$ of $L^\infty$ (so $C_k^{\text{cell}}(L^\infty) = H_k^{\text{sing}}(L^k, L^{k-1})$, and the boundary map $\partial^{\text{cell}} : C_k^{\text{cell}}(L^\infty) \to C_{k-1}^{\text{cell}}(L^\infty)$ is the composition $i_* d$, where $i : (L^{k-1}, \emptyset) \hookrightarrow (L^k, L^{k-1})$ is inclusion and $d$ is the connecting homomorphism arising from the long exact sequence of the pair $(L^k, L^{k-1})$) then

$$H_*^{\text{cell}}(L^\infty)) \cong H_*^{\text{sing}}(L^\infty, L^{-1}) = H_*^{\text{sing}}(L^\infty) \cong H_*^{\text{sing}}(\mathcal{L}),$$

the last isomorphism following from Proposition 2.16.

Let $CM_k(X)$ denote the free abelian group generated by the rest points of $X$ of Morse index $k$. The Morse Homology Theorem is now an easy corollary of Proposition 2.17.

**Corollary 2.18. (The Morse Homology Theorem)**
*There exists an isomorphism*

$$CM_k(X) \cong C_k^{\text{cell}}(L^\infty).$$

*Proof.* We carry on from the proof of Proposition 2.17, using the same notation:

$$\begin{aligned} C_k^{\text{cell}}(L) = H_k^{\text{sing}}(L^k, L^{k-1}) &\cong \bigoplus_{x\in\text{rest}_k(X)} H_k^{\text{sing}}(U(x), U(x) \cap L^{k-1}) \\ &\cong \bigoplus_{x\text{rest}_k(X)} H_k^{\text{sing}}(D^k, S^{k-1}) \\ &\cong \bigoplus_{x\in\text{rest}_k(X)} \mathbb{Z} = CM_k(X). \end{aligned}$$

■

Now we state a theorem that shows we can define the boundary homomorphism $\partial^M$ geometrically in terms of the intersection numbers of stable and unstable manifolds of rest points of Morse index differing by 1.

Suppose $m(x) = m(y) + 1$. Then $\mathcal{W}(x, y)$ has dimension 1, and hence each component of $\mathcal{W}(x, y)$ is a line. Let $m(x, y)$ denote the number of connected components of $\mathcal{W}(x, y)$, taken mod 2. Note that the second statement in Proposition 2.12 shows this is only a finite sum. The result is then the following.

**Theorem 2.19. (The Morse Boundary Homomorphism Theorem)**
*We may compute the boundary homomorphism $\partial^M : CM_k(X) \to CM_{k-1}(X)$ alternatively by*

$$\partial^M(x) = \sum_{y\in rest_{k-1}(X)} m(x, y)y. \tag{2.3}$$

Proofs of this result can be found in [AM], Theorem 2.11 and [BH], Theorem 7.4. We remark that a special case of the **$\lambda$-lemma**, another crucial result from the theory of smooth dynamical systems is needed for the proof.

Finally, let us clear two points of contention that have arisen. Firstly our definition of the Morse complex appears to depend on the choice of filtration function. This is in fact the case, and one way to get round this is to define the Morse complex of $X$ to be a suitable direct limit of complexes $C_*^{\text{cell}}(L^\infty(F))$. That this is possible and well defined is the subject of [AM], Theorem 2.8(iii), to which we direct the reader.

Secondly we describe how one would go about removing the assumption that only there are only finitely many rest points of a given degree. Without this assumption there does not necessarily exist a filtration function, which as we have seen, is crucial for associating a cellular filtration of $\mathcal{L}$ with $X$. It turns out the correct way to proceed is to set $\mathcal{L}^a := \{x \in \mathcal{L} \mid f(x) < a\}$ and then to take the direct limit of the complexes $C_*^{\text{cell}}(X, \mathcal{L}^a)$ as $a \to \infty$, for Lemma 2.4 ensures that on each $\mathcal{L}^a$, the finiteness condition is satisfied. More details can be found in [AM], §2.9.

## 3. Hamiltonians and Lagrangians

In this section we discuss the essential background knowlege we will need in the next section to construct the Floer complex. I have consistently taken a 'low-tech' approach (similar to Appendix B of [W1]) and have favoured explicit computations in local coordinates over the more technically demanding coordinate-free approach given in say, Chapter III, §2.2 of [AL] or §1.K of [Bes].

### The cotangent bundle

We being this section with a summary of results we will need about the cotangent bundle $T^*M$. Recall that $g$ is a fixed Riemannian metric on $M$. Let $\{\Gamma_{ij}^k\}$ denote the Christoffel symbols of the Levi-Civita connection associated to $g$. Let $\tau : T^*M \to M$ and $\pi : TT^*M \to T^*M$ denote the footpoint maps.

Suppose $(x^i) = (x^1, \dots, x^n)$ are local coordinates on $U \subseteq M$. If $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is the chart, we obtain coordinates on $T^*U$ via

$$(D\phi)^{*-1} : T^*U \to \phi(U) \times \mathbb{R}^n,$$

which maps

$$(x, p) \mapsto \left(\phi(x), (D\phi_x)^{*-1}(p)\right).$$

We let $(x^i, p_j) = (x^1, \dots, x^n, p_1, \dots, p_n)$ denote these coordinates. We can repeat the process to obtain coordinates on $TT^*U$: if $\psi = (D\phi)^{*-1}$ then we obtain

$$(D\psi)^{*-1} : TT^*U \to (\phi(U) \times \mathbb{R}^n) \times T(\phi(U) \times \mathbb{R}^n).$$

We let $(x^i, p_j, y^k, q_\ell)$ denote these coordinates. Unfortunately the $(q_\ell)$ **do not** transform as the coordinate functions of a covector, and thus we cannot simply obtain a natural isomorphism

$$TT^*M \to TM \oplus T^*M$$
$$\left(x^i, p_j, y^k, q_\ell\right) \mapsto \left(x^i, y^k\right) \oplus \left(x^i, q_\ell\right).$$

This is where the metric comes in: it turns out that if $h_\ell := q_\ell - \Gamma_{\ell k}^j y^k p_j$ then the $(h_\ell)$ do transform as the coordinate functions of a covector - see [W1], §B.1.1[2].

Thus we are led to a vector bundle isomorphism between the bundle $\pi : TT^*M \to T^*M$ and the pullback bundle $\text{pr}_1 : \tau^*(TM \oplus T^*M) \to T^*M$ defined fibrewise by

$$T_{(x,p)}T^*M \to \{(x, p)\} \oplus T_xM \oplus T_x^*M \tag{3.1}$$
$$(x^i, p_j, y^k, q_\ell) \mapsto \left(x^i, p_j\right) \oplus \left(x^i, y^k\right) \oplus \left(x^i, q_\ell - \Gamma_{\ell k}^j y^k p_j\right).$$

We shall generally regard this as an isomorphism $T_{(x,p)}T^*M \to T_xM \oplus T_x^*M$, and we call it the **natural isomorphism** (since the $(h_\ell)$ transform correctly this is indeed coordinate independent) between $TT^*M$ and $TM \oplus T^*M$. For convenience we write

$$\xi \doteqdot (y, h) \tag{3.2}$$

to indicate that $\xi \in T_{(x,p)}T^*M$ corresponds to $(y, h) \in T_xM \oplus T_x^*M$ under the natural isomorphism.

**Lemma 3.1.** *Let $a(t) = (x(t), p(t))$ be a loop on $T^*M$. Then under the natural isomorphism, $\dot{a}(t) \doteqdot \left(\dot{x}(t), \nabla_{\dot{x}}^* a(t)\right)$, where $\nabla^*$ denotes the Levi-Civita connection on $M$ acting on covector fields.*

*Proof.* Taking local coordinates $x^i(t)$ about $x(t)$ (that is, if $(x^1, \dots, x^n)$ are coordinates on a neighborhood $U$ of $x(t)$ and $x^i(t) := x^i \circ x(t)$), we can write

$$a(t) = p_j(t)dx^j(t),$$

where $p = (p_1, \dots, p_n) : S^1 \to \mathbb{R}^n$ is a smooth curve. Then we have

$$\dot{a}(t) = \dot{x}^k(t)\partial_k(t) + \dot{p}_\ell(t)e_\ell,$$

where the $e_\ell$ are the standard basis vectors of $\mathbb{R}^n$, and by $\partial_k(t)$ we mean $\left.\frac{\partial}{\partial x^k}\right|_{x(t)}$.

Hence $\dot{x}(t) = \dot{x}^k(t)\partial_k(t)$ and

$$\begin{aligned} \nabla_{\dot{x}}^* a(\partial_\ell) &= \nabla_{\dot{x}^k\partial_k}^* \left(p_j dx^j\right)(\partial_\ell) \\ &= \dot{x}^k \nabla_{\partial_k}\left(p_j dx^j(\partial_\ell)\right) - p_j \dot{x}^k dx^j(\nabla_{\partial_k}\partial_\ell) \\ &= \dot{x}^k \nabla_{\partial_k}\left(p_j \delta_\ell^j\right) - \dot{x}^k p_j \Gamma_{k\ell}^j \\ &= \dot{p}_\ell - \dot{x}^k p_j \Gamma_{k\ell}^\ell. \end{aligned} \tag{3.3}$$

■

[2]The motivation for the choosing the $h_\ell$ is that if $K : TT^*M \to T^*M$ denotes the **connection map** then in local coordinates $K(x^i, p_j, y^k, q_\ell) = (x^i, q_\ell - \Gamma_{\ell k}^j y^k p_j)$.

**Definition 3.2.** The **Liouville 1-form** is a 1-form $\theta$ on $T^*M$ defined fibrewise by

$$\theta_{(x,p)} : T_{(x,p)}T^*M \to \mathbb{R}$$
$$\xi \mapsto \pi(\xi)\left(D\tau_{(x,p)}(\xi)\right).$$

In local coordinates $(x^i, p_j, y^k, q_\ell)$ on $TT^*M$, we have

$$\theta : \left(x^i, p_j, y^k, q_\ell\right) \mapsto \left(x^i, p_j dx^i\right)\left(x^i, y^k\partial_{x^k}\right) = p_j y^k dx^j\,(\partial_{x^k}) = p_j y^j,$$

and hence $\theta(x^i, p_j) = p_j dx^j$, so

$$\theta\left(y^i\partial_{x^i} + q_j\partial_{p_j}\right) = p_j y^j. \tag{3.4}$$

Let $a : S^1 \to T^*M$ be a curve on $T^*M$. The following observation will prove useful later: writing $\dot{a}(t) = y^i\partial_{x^i} + q_j\partial_{p_j}$ in local coordinates $(x^i(t), p_j(t), y^k(t), q_\ell(t))$ on $TT^*M$, we have by (3.4) that

$$\theta(\dot{a}) = p_j y^j.$$

Now note that $\dot{x}(t) = y^k\partial_{x^k}$ and $p(t) = p_j dx^j$ and thus $p(t)(\dot{x}(t)) = p_j dx^j(y^k\partial_{x^k}) = p_j y^j$, and thus we have deduced the following:

$$a^*(\theta) = p(\dot{x}). \tag{3.5}$$

**Definition 3.3.** A **symplectic form** $\omega$ on a manifold $N$ is a closed 2-form such that the skew symmetric bilinear form $\omega_x : T_xN \times T_xN \to \mathbb{R}$ is non-degenerate for all $x \in N$. A **symplectic manifold** $(N, \omega)$ is a manifold equipped with a symplectic form $\omega$.

It is not hard to see that $T^*M$ can naturally be given the structure of a symplectic manifold. Indeed, define a 2-form $\omega$ by $\omega = d\theta$. $\omega$ is certainly closed. In local coordinates $(x^i, p_j)$ we have $\omega(x^i, p_j)(\cdot,\cdot) = (dp_i \wedge dx^i)(\cdot,\cdot)$; since $\{dx^i, dp_i\}_{i=1}^n$ is a basis of $T^*_{(x,p)}T^*M$, it follows $\omega$ is non-degenerate. We call $\omega$ the **canonical symplectic form** on $T^*M$.

Under the natural isomorphism (3.1), if $\xi \doteqdot (y, h)$ and $\eta \doteqdot (z, f)$ then an easy check in local coordinates gives

$$\omega(\xi, \eta) = h(z) - f(y). \tag{3.6}$$

In local coordinates the metric $g$ is given by $n^2$ functions $(g_{ij})$. The **dual metric** $g^* = \langle\cdot,\cdot\rangle^*$ is[3] given locally by the $n^2$ functions $(g^{k\ell})$ where $g^{k\ell}g_{\ell i} = \delta^k_i$. We can define a metric $G$ on $TT^*M$ as follows: if as before $\xi \doteqdot (y, h)$ and $\eta \doteqdot (z, f)$ then

$$G(\xi, \eta) = \langle y, z\rangle + \langle h, f\rangle^*.$$

$G$ is clearly symmetric, positive definite and non degenerate[4].
Recall that an **almost complex structure** $J$ on $T^*M$ is an endomorphism $TT^*M \to TT^*M$ such that $J^2 = -\mathbb{1}$. The metric $g$ on $M$ determines a canonical almost complex structure $J_g$ on $TT^*M$ given under

$$J_g(\xi) = J_g(y, h) = \left(h^\#, -y^\flat\right),$$

where $\flat : T_xM \to T^*_xM$ is the (musical) isomorphism induced by the metric, that is, $v^\flat(w) = \langle v, w\rangle$ for $v, w \in T_xM$ and $\# : T^*_xM \to T_xM$ is its inverse. An almost complex structure $J$ on $TT^*M$ is called **compatible** with $\omega$ if

$$g_J = \langle\cdot,\cdot\rangle_J := \omega(\cdot, J\cdot)$$

defines a Riemannian metric on $T^*M$.

An important fact is that $J_g$ is compatible with $\omega$; in fact $\langle\cdot,\cdot\rangle_{J_g}$ coincides with the metric $G$ introduced earlier, since if $\xi \doteqdot (y, h)$ and $\eta \doteqdot (z, f)$ as before then

$$\omega\left(\xi, J_g(\eta)\right) = h\left(f^\#\right) - \left(-z^\flat\right)(y) = \langle h, f\rangle^* + \langle y, z\rangle = G(\xi, \eta). \tag{3.8}$$

Note also that

$$\omega\left(J_g(\xi), J_g(\eta)\right) = \omega(\xi, \eta). \tag{3.9}$$

[3] Where no confusion is possible I shall drop the '$*$' and refer to both metrics as $\langle\cdot,\cdot\rangle$.

[4] Another way of writing $G$ would be to use the connection map $K : TT^*M \to T^*M$; then $G$ is defined by

$$G(\xi, \eta) := \langle D\pi(\xi), D\pi(\eta)\rangle + \langle K(\xi), K(\eta)\rangle^*, \tag{3.7}$$

This metric is called the **Sasaki metric** (actually strictly speaking, the name Sasaki metric is normally used for the equivalent construction on the tangent bundle, not the cotangent bundle).

**The Legendre transformation**

We are interested in time dependent 1-periodic **Hamiltonians** on $T^*M$ of the form kinetic plus potential energy. By this we mean a smooth function

$$H : S^1 \times T^*M \to \mathbb{R}$$
$$(t, x, p) \mapsto \frac{1}{2}\langle p, p\rangle^* + V(t, x), \tag{3.10}$$

where $V \in C^\infty(S^1 \times M, \mathbb{R})$ is a potential function. Under the **Legendre condition**

$$\det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) \neq 0$$

(which in this case is satisfied since $\det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) = \det\left(g^{ij}\right)$, where $(x^i, p_j)$ are coordinates on $T^*M$), the **Legendre transformation**[5] is defined globally, and we obtain a time dependent **Lagrangian** $L : S^1 \times TM \to \mathbb{R}$ from $H$ defined by

$$L(t, x, v) := \ell(x, v)(v) - H(t, x, \ell(x, v)),$$

where if $v = v^i \partial_i$ then $\ell(x, v) \in T_x^* M$ is given by $\ell(x, v) = p_j dx^j$ with the $(p_j)$ defined by

$$v^i = \frac{\partial H}{\partial p_i} = g^{ij} p_j,$$

that is,

$$p_j = g_{ij} v^i.$$

Thus

$$\begin{aligned} L(t, x, v) &= p_i v^i - \frac{1}{2} g^{ij} p_i p_j - V(t, x) \\ &= \frac{1}{2}\langle v, v\rangle - V(t, x). \end{aligned} \tag{3.11}$$

There is a uniquely determined time dependent vector field $X$ on $T^*M$ arising from $H$ defined by

$$\omega(X(t, x, p), \xi) = -dH(t, x, p)(\xi), \qquad \xi \in T_{(x,p)}T^*M.$$

Note that $X = J_g \nabla H$, where $\nabla H$ denotes the gradient of $H$ with respect to $G$, since

$$\begin{aligned} \omega(X, \xi) &= -dH(\xi) \\ &= -G(\nabla H, \xi) \\ &= -\omega\left(\nabla H, J_g(\xi)\right) \\ &= -\omega\left(J_g \nabla H, J_g^2(\xi)\right) \\ &= -\omega\left(J_g \nabla H, -\xi\right). \end{aligned}$$

The flow of $X$ is denoted by $\varphi^t$. Physically, $\varphi^t$ can be thought of representing the motion of a charge on $M$ moving under the effect of the (electrostatic) potential $V$.

Let $\mathcal{P}(H)$ denote the set of solutions $a \in C^\infty(S^1, T^*M)$ of the initial value problem

$$X(t, a(t)) = \dot{a}(t), \qquad a(0) = a_0. \tag{3.12}$$

The **action functional** associated to $H$ is defined by

$$\begin{aligned} \mathcal{A}(a) &= \int_{S^1} a^*(\theta) - \int_0^1 H(t, a(t)), \qquad \text{for } a : S^1 \to T^*M, \\ &= \int_0^1 p(t)(\dot{x}(t)) - H(t, x(t), p(t))dt, \qquad a(t) = (x(t), p(t)). \end{aligned} \tag{3.13}$$

where $\theta$ is the Liouville 1-form (we are using (3.5) here).

Let $\mathrm{crit}(\mathcal{A})$ denote the set of critical points of $\mathcal{A}$. The next result shows that $\mathcal{P}(H) \subseteq \mathrm{crit}(\mathcal{A})$.

[5]More precisely, this is actually the inverse Legendre transformation: the Legendre transformation converts Lagrangians to Hamiltonians. For more information see [MS2], §1.1.

**Proposition 3.4.** *For $a \in C^\infty(S^1, T^*M)$ and $\xi$ a section of the pullback bundle $a(TT^*M)$, we have*

$$d\mathcal{A}_a(\xi) = \int_0^1 \omega\left(\xi(t), \dot a(t) - X(t, a(t))\right) dt. \tag{3.14}$$

*Proof.* We work locally; write $a(t) = \left(x^i(t), p_j(t)\right)$ and $\xi(t) = (x^i(t), p_j(t), y^k(t), q_\ell(t))$. Now consider a variation $a_s(t) = (x^i_s(t), p^s_j(t))$ such that $a_0 = a$ and $\frac{d}{ds}\Big|_{s=0}(x^i_s, p^s_j) = (y^i, q_\ell)$. Then

$$\begin{aligned} d\mathcal{A}_a(\xi) &= \frac{d}{ds}\Big|_{s=0}\mathcal{A}\left(x^i_s, p^s_j\right) \\ &= \frac{d}{ds}\Big|_{s=0}\int_0^1 p^s_j \dot x^j_s - \frac{1}{2}g^{ij}(x_s)p^s_i p^s_j - V(t, x^i_s) \qquad (3.15)\\ &= \int_0^1 \left(q_j\left(\dot x^j - \frac{1}{2}g^{ij}(x)p_i\right) + p_j\left(\dot y^j - \frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x^k}y^k p_i - \frac{1}{2}g^{ij}(x)q_i\right) - \frac{\partial V(t,x)}{\partial x^k}y^k\right) dt. \end{aligned}$$

As before we wish to work with the intrinsic quantity $h_\ell = q_\ell + \Gamma^j_{\ell k}p_j y^k$. Replacing $q_\ell$ with $h_\ell$ in the above formula, and performing simple computations we obtain

$$\begin{aligned} d\mathcal{A}(a)(\xi) &= \int_0^1 \Bigg[h_j\dot x^j - g^{ij}(x)p_i h_j - \left(\dot p_k - \Gamma^\ell_{jk}(x)p_\ell \dot x^j\right)y^k + \left(\frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x^k} - g^{i\ell}(x)\Gamma^j_{i\ell}(x)\right)p_\ell y^k \\ &\quad - \frac{\partial V(t,x)}{\partial x^k}y^k\Bigg] dt \\ &= \int_0^1 h(\dot x) - \langle h, p\rangle^* - \nabla^*_{\dot x}a(y) - dV(y)dt, \qquad (3.16) \end{aligned}$$

where we are using (3.3) and the equality

$$\sum_{i,j,k}\left(\frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x^k} - g^{i\ell}(x)\Gamma^j_{i\ell}(x)\right) = 0. \tag{3.17}$$

Verfiying (3.17) is painful; for completeness the proof is included[6], but is relegated to Appendix A.

Now we have

$$\omega(\xi, \dot a) = h(\dot x) - \nabla^*_{\dot x}a(y), \tag{3.18}$$

and we claim

$$dH(\xi) = dV(y) + \langle h, p\rangle^*. \tag{3.19}$$

Given this, (3.16), (3.18) and (3.19) immediately imply the result.

To verify (3.19) we compute:

$$\begin{aligned} dH(\xi) &= \frac{\partial H}{\partial x^k}\left(t, x^i, p_j\right)y^k + \frac{\partial H}{\partial p_\ell}\left(t, x^i, p_j\right)q_\ell \\ &= \frac{\partial V}{\partial x^k}\left(t, x^i\right)y^k + \frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x^k}p_i p_j y^k + g^{ij}(x)p_i q_j \\ &= \frac{\partial V}{\partial x^k}\left(t, x^i\right)y^k + \frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x^k}p_i p_j y^k + g^{i\ell}(x)p_i\left(h_\ell + \Gamma^j_{\ell k}(x)y^k p_j\right) \\ &= \frac{\partial V}{\partial x^k}\left(t, x^i\right)y^k + \left(\frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x^k} - g^{i\ell}(x)\Gamma^j_{i\ell}(x)\right)p_i p_j y^k + g^{i\ell}(x)p_i h_\ell \\ &= dV(y) + \langle p, h\rangle^*, \end{aligned}$$

where in the last but one line we used (3.17) again. ■

Note that this also shows that

$$X(t, x, p) \doteqdot \left(p^\#, -dV_{(t,x)}\right). \tag{3.20}$$

To obtain equality crit$(\mathcal{A}) = \mathcal{P}(H)$ we must extend the domain of $\mathcal{A}$ and work in the free loop space instead. A good reference for the statements in the next paragraph is [K], §2.3 and §2.4.

[6] I failed to find a satisfactory reference.

Given a manifold $N$ (in this paper we are interested in the case $N = M$ or $N = T^*M$) the **free loop space** $\mathcal{L}N$ is the set of loops on $N$ of Sobolev class $W^{1,2}$ (that is, their expression in local coordinates is of class $W^{1,2}$). $\mathcal{L}N$ carries the structure of a Hilbert manifold, and given $a \in \mathcal{L}N$ the tangent space $T_a\mathcal{L}N$ is in 1-1 correspondence with the $W^{1,2}$ vector fields along $a$. If $g$ is a Riemannian metric on $N$ then $g$ determines a complete Riemannian metric $\mathcal{G}_g$ on $\mathcal{L}N$ given by

$$\mathcal{G}_g(\xi(t), \eta(t)) := \int_0^1 \langle \xi(t), \eta(t) \rangle + \langle \nabla_{\dot{x}}\xi, \nabla_{\dot{x}}\eta \rangle \, dt, \qquad \xi(t), \eta(t) \in T_a\mathcal{L}N.$$

*Remark.* The inclusion $\mathcal{L}N \hookrightarrow C^0(S^1, N)$ is actually a homotopy equivalence and hence $H_*^{\text{sing}}(\mathcal{L}N) \cong H_*^{\text{sing}}\left(C^0(S^1, N)\right)$; this will ultimately allow us to deduce that the Floer homology of the cotangent bundle is isomorphic to the singular homology of the space of **all** (continuous) loops on $M$. For the sake of clarity however we won't actually use this observation in this paper.

Returning back to the situation at hand and taking $N = T^*M$, since elements $a \in \mathcal{L}T^*M$ are almost everywhere differentiable, the definition of $\mathcal{A}$ is still meaningful on $\mathcal{L}T^*M$, and thus we can extend $\mathcal{A}$ to a functional on $\mathcal{L}T^*M$. Using a simple argument involving approximation by smooth functions we see that (3.14) still holds on $\mathcal{L}T^*M$. Standard regularity results then imply that any critical point of $\mathcal{A}$ is smooth, and hence (3.14) implies that $\text{crit}(\mathcal{A}) = \mathcal{P}(H)$.

The Lagrangian $L$ also determines a functional $\mathcal{S}$, this time on the free loop space $\mathcal{L}M$ of $M$, defined by

$$\mathcal{S}(x) = \int_0^1 L(t, x(t), \dot{x}(t))\, dt = \int_0^1 \left( \frac{1}{2}|\dot{x}(t)|^2 - V(t, x(t)) \right) dt, \qquad \text{for } x : S^1 \to M.$$

Let $\mathcal{P}(L)$ denote the set of critical points of $\mathcal{S}$. The following result is proved in a similar fashion to Proposition 3.4. In fact this result is considerably more standard; it follows from the **first variation formula** and its proof can be found in any standard textbook on Riemannian geometry; see for instance [J], §4.1.

**Proposition 3.5.** *An element $x \in \mathcal{L}M$ is an element of $\mathcal{P}(L)$ if and only if*

$$-\nabla_{\dot{x}}\dot{x}(t) - \nabla V(t, x(t)) = 0.$$

This lets us deduce:

**Corollary 3.6.** *An element $a \in \mathcal{L}T^*M$ is an element of $\mathcal{P}(H)$ if and only if, writing $a(t) = (x(t), p(t))$, we have $x(t) \in \mathcal{P}(L)$ and*

$$p(t) = \dot{x}(t)^\flat.$$

*Proof.* By Proposition 3.7 and (3.20) the assertion that $x \in \mathcal{P}(L)$ and $p = \dot{x}^\flat$ are equivalent to the fact that $\dot{a}(t) = X(t, a(t))$, which by Proposition 3.4 is equivalent to $a \in \mathcal{P}(H)$. ∎

## Magnetic fields

Now we introduce the other type of symplectic form we will study on $T^*M$. Let $\sigma$ be a closed 2-form on $M$. Then we can consider the form $\Omega := \omega + \tau^*\sigma$. We say that $\Omega$ is a **twisted symplectic form**; it is certainly closed, and since $\tau^*\sigma$ vanishes on the fibres of the projection it is easily seen that $\Omega$ is non-degenerate. Explicitly if $\xi \doteqdot (y, h)$, $\eta \doteqdot (z, f)$ then

$$\begin{aligned} \Omega(\xi, \eta) &= \omega(\xi, \eta) + \tau^*\sigma(\xi, \eta) \\ &= h(z) - f(y) + \sigma(y, z), \end{aligned}$$

whence it is clear that if $\xi \neq 0$ we can choose $\eta$ such that $\Omega(\xi, \eta) \neq 0$; if $h \neq 0$ take $\eta = (y, f)$ where $h(y) \neq f(y)$, and if $h = 0$ take $\eta = (0, f)$ where $f(y) \neq 0$.

The motivation behind this construction is from classical mechanics for which we refer the reader to [G]; essentially the Hamiltonian flow of $\Omega$ provides a mathematical model for the motion of a charge in a magnetic field, represented by $\sigma$.

We wish to establish a result similar to Proposition 3.4. To do this we need to assume $\sigma$ is exact, say $\sigma = d\alpha$[7]. Since then $\tau^*\sigma = \tau^*(d\alpha) = d(\tau^*\alpha)$ we can then introduce the **twisted Liouville form** $\Theta = \theta + \tau^*\alpha$,

[7] It is an extremely interesting (and to some extent, I believe, open) question as to how much of the essay would go through if we didn't make this assumption. A natural compromise would be to assume that $\sigma$ is **weakly exact**; by this we mean that $\sigma|_{\pi_2(M)} = 0$. This is required in order to ensure the precompactness results (Theorem 4.12) go through - see for instance [Lau], §4.

so $\Omega = d\Theta$. Next we define the **twisted action functional**,

$$\mathcal{A}^{\text{tw}}(a) : \;=\; \int_{S^1} a^*(\Theta) - \int_0^1 H(t, a(t)), \qquad \text{for } a : S^1 \to T^*M.$$

The goal then is to obtain a vector field $Y$ such that the equivalent of Proposition 3.4 holds, namely:

**Proposition 3.7.** *If $Y$ is the vector field such that*

$$\Omega(Y, \cdot) = -dH(\cdot)$$

*then*

$$d\mathcal{A}^{tw}(a)(\xi) = \int_0^1 \Omega\left(\xi(t), \dot{a}(t) - Y(t, a(t))\right) dt. \tag{3.21}$$

*Proof.* This is in fact immediate from Proposition 3.4 and the construction of $\Omega$ and $\mathcal{A}^{\text{tw}}$ but it is satisfying to prove it directly. It will also be helpful to get an explicit description of the the vector field $Y$. For this, define the **Lorentz map** $\mathscr{L} : TM \to TM$ associated to $\sigma$ by

$$\sigma(u, w) = \langle \mathscr{L}(u), w \rangle .$$

Now suppose $Y \doteqdot (Y_1, Y_2)$. Then we must solve

$$\Omega(Y, \xi) = \omega(Y, \xi) + \tau^*\sigma(Y, \xi) = -dH(\xi),$$

that is,

$$Y_2(y) - h(Y_1) + \langle \mathscr{L}(Y_1), y \rangle = -dV(y) - \langle p, h \rangle^*$$

It is then clear that

$$Y \doteqdot \left(p^\#, -dV - \left(\mathscr{L}\left(p^\#\right)\right)^\flat\right). \tag{3.22}$$

Now let us verify (3.21). We proceed as before: work locally and write $a(t) = \left(x^i(t), p_j(t)\right)$ and $\xi(t) = (x^i(t), p_j(t), y^k(t), q_\ell(t))$. Now consider a variation $a_s(t) = (x^i_s(t), p^s_j(t))$ such that $a_0 = a$ and $\frac{d}{ds}\Big|_{s=0}(x^i_s, p^s_j) = (y^i, q_\ell)$. Then

$$\begin{aligned} d\mathcal{A}^{\text{tw}}(a)(\xi) &= \frac{d}{ds}\Big|_{s=0}\mathcal{A}^{\text{tw}}\left(x^i_s, p^s_j\right) \\ &= \frac{d}{ds}\Big|_{s=0}\int_0^1 \alpha_j(x_s)\dot{x}^j_s + p^s_j\dot{x}^j_s - \frac{1}{2}g^{ij}(x_s)p^s_i p^s_j - V(t, x^i_s) \\ &= \int_0^1 \alpha_j(x)\dot{y}^j + \frac{\partial\alpha_j(x)}{\partial x^k}y^k\dot{x}^j dt + d\mathcal{A}(a)(\xi) \\ &= \int_0^1 -\dot{\alpha}_j(x)y^j + \frac{\partial\alpha_j(x)}{\partial x^k}y^k\dot{x}^j dt + d\mathcal{A}(a)(\xi) \\ &= \int_0^1 -\frac{\partial\alpha_j(x)}{\partial x^k}\dot{x}^k y^j + \frac{\partial\alpha_j(x)}{\partial x^k}y^k\dot{x}^j dt + d\mathcal{A}(a)(\xi) \\ &= \int_0^1 d\alpha\,(y, \dot{x}) + h\,(\dot{x}) - \langle h, p\rangle^* - \nabla^*_{\dot{x}}a(y) - dV(t, x)(y)dt, \end{aligned} \tag{3.23}$$

where we are using on the last line the easily verified fact that if $V = v^i\partial_i$ and $W = w^j\partial_j$ are vector fields and $\omega = \omega_k dx^k$ is a 1-form then

$$d\omega(V, W) = v^i w^j\left(\frac{\partial\omega_j}{\partial x^i} - \frac{\partial\omega_i}{\partial x^j}\right).$$

Now one simply notes that

$$\begin{aligned} \Omega(\xi, \dot{a} - Y(a)) &= \omega\,(\xi, \dot{a}) + \tau^*\sigma(\xi, \dot{a}) - \Omega(\xi, Y(a)) \\ &= h(\dot{x}) - \nabla^*_{\dot{x}}a(y) + d\alpha(y, \dot{x}) - dH(\xi), \end{aligned}$$

and thus it is immediate from (3.22) that (3.21) is satisfied. ∎

It follows if we let $\mathcal{P}_{\mathrm{tw}}(H)$ denote the set of smooth loops $a : S^1 \to T^*M$ such that $Y(t, a(t)) = \dot{a}(t)$ then the critical points of $\mathcal{A}^{\mathrm{tw}}$ are precisely the elements of $\mathcal{P}_{\mathrm{tw}}(H)$.

We now want to deduce an analogous assertion to Corollary 3.6 for the twisted symplectic form $\Omega$. In other words, we want to define a new functional $\mathcal{S}^{\mathrm{tw}} : \mathcal{L}M \to \mathbb{R}$ such that $a : S^1 \to T^*M$ is a critical point of $\mathcal{A}^{\mathrm{tw}}$ if and only if, writing $a(t) = (x(t), p(t))$ we have $x$ a critical point of $\mathcal{S}^{\mathrm{tw}}$ and $p(t) = \dot{x}(t)^\flat$.

The trick to this is guessing what to take as our function $\mathcal{S}^{\mathrm{tw}}$. The key is provided by the computation in (3.23); we showed here that if $x_s(t)$ is a variation such that $x_0 = x$ and $\frac{d}{ds}\big|_{s=0} x_s^i = y^i$, then

$$\frac{d}{ds}\Big|_{s=0} \int_0^1 \alpha_i(x_s)\dot{x}_s^i dt = \int_0^1 d\alpha(y, \dot{x})dt.$$

It follows that if we define

$$L^{\mathrm{tw}} : TM \to \mathbb{R}, \qquad (t, x, v) \mapsto \frac{1}{2}|v|^2 - \alpha_x(v) - V(t, x), \tag{3.24}$$

and then set

$$\mathcal{S}^{\mathrm{tw}}(x) := \int_0^1 L^{\mathrm{tw}}(t, x(t), \dot{x}(t))dt$$

then $x \in \mathcal{L}M$ is a critical point of $\mathcal{S}^{\mathrm{tw}}$ if and only if

$$-\nabla_{\dot{x}}\dot{x}(t) - \nabla V(t, x(t)) - \mathscr{L}(\dot{x})^\flat = 0,$$

and then we have the desired result:

**Proposition 3.8.** *An element $a \in \mathcal{L}T^*M$ is an element of $\mathcal{P}_{tw}(H)$ if and only if, writing $a(t) = (x(t), p(t))$, we have $x(t) \in \mathcal{P}(L^{tw})$ and*

$$p(t) = \dot{x}(t)^\flat.$$

This suggests an alternative way of looking at the problem, for which we need to compute the Legendre transformation of $L^{\mathrm{tw}}$.

**Lemma 3.9.** *The Legendre transformation of $L^{\mathrm{tw}}$ is the Hamiltonian*

$$H^{\mathrm{tw}} : T^*M \to \mathbb{R}, \qquad (t, x, p) \mapsto \frac{1}{2}|p + \alpha_x|^2 + V(t, x).$$

*Proof.* In order to parallel the method we used before we shall in fact show that the (inverse) Legendre transformation of $H^{\mathrm{tw}}$ is $L^{\mathrm{tw}}$. We note that the Legendre condition is still satisfied, since

$$\det\left(\frac{\partial^2 H^{\mathbf{tw}}}{\partial p_i \partial p_j}\right) = \det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) = \det\left(g^{ij}\right) \neq 0.$$

Thus as before the (inverse) Legendre transformation is defined globally, and we obtain a time dependent Lagrangian (temporarily denoted by) $\tilde{L} : S^1 \times TM \to \mathbb{R}$ from $H$ defined by

$$\tilde{L}(t, x, v) := \ell(x, v)(v) - H(t, x, \ell(x, v)),$$

where if $v = v^i\partial_i$ then $\ell(x, v) \in T_x^*M$ is given by $\ell(x, v) = p_j dx^j$ with the $(p_j)$ defined by

$$v^i = \frac{\partial H}{\partial p_i} = g^{ij}\left(p_j + \alpha_j\right),$$

that is,

$$p_j = g_{ij}v^i - \alpha_j.$$

Hence

$$\begin{aligned}
\tilde{L}(t, x, v) &= p_j v^j - \frac{1}{2} g^{ij}(p_i + \alpha_i)\left(p_j + \alpha_j\right) - V(t, x) \\
&= g_{ij}v^i v^j - \alpha_j v^j - \frac{1}{2} g_{ij}v^i v^j - V(t, x) \\
&= \frac{1}{2}\langle v, v\rangle - \alpha_x(v) - V(t, x) \\
&= L^{\mathrm{tw}}(t, x, v).
\end{aligned}$$

■

Running the entire argument backwards we deduce the following:

**Proposition 3.10.** *We have*

$$\mathcal{P}_{\mathrm{tw}}(H) \cong \mathcal{P}\left(H^{\mathrm{tw}}\right), \tag{3.25}$$

*where the isomorphism is given as follows: given $x \in \mathcal{P}(L^{\mathrm{tw}})$ there exists a unique $a \in \mathcal{P}_{\mathrm{tw}}(H)$, where, writing $a(t) = (x(t), p(t))$ then $p(t) = \dot{x}(t)^\flat$, and similarly there exists a unique $b \in \mathcal{P}(H^{\mathrm{tw}})$ where, writing $b(t) = (x(t), r(t))$ then $r(t) + \alpha_{x(t)} = \dot{x}(t)^\flat$.*

*Proof.* The only thing that needs proving is the assertion that $r(t) + \alpha_{x(t)} = \dot{x}(t)^\flat$. A simple computation in local coordinates akin to Proposition 3.5 yields this easily. ∎

*Remark.* Proposition 3.10 has essentially given a way to eliminate $\Omega$ from the calculations. This will be very helpful when we come to set up the Floer complex in the next section, which, like the Morse complex of §2 will be defined by counting critical points, and thus the complex generated by the elements of $\mathcal{P}_{\mathrm{tw}}(H)$ is canonically isomorphic to that generated by $\mathcal{P}(H^{\mathrm{tw}})$. As mentioned in the introduction, we are following the approach of Abbondandolo and Schwarz in [AS]; they set up the Floer complex $HF_*(T^*M, \omega, H, J)$ for Hamiltonians satisfying certain decay conditions, and both $H$ and $H^{\mathrm{tw}}$ as defined in this paper satisfy these conditions. Thus we will deduce the corresponding results for $HF_*(T^*M, \Omega, H) \cong HF_*(T^*M, \omega, H^{\mathrm{tw}})$ 'for free'.

## Applying the Morse homology theorem

In what follows we shall almost exclusively work the Hamiltonian $H$ rather than $H^{\mathrm{tw}}$, since the formulae are simpler here, and in almost all cases it is clear that the results carry over verbatim to the twisted case. We shall outline the places where this is not immediate.

It is not possible to do Morse theory with the functional $\mathcal{A}$; the critical points of $\mathcal{A}$ need not have finite Morse index, and worse, $\mathcal{A}$ does not even define a flow (see [Sa])! $\mathcal{S}$ however is much better behaved. Let $\mathcal{X} = -\nabla\mathcal{S}$ denote the negative gradient vector field of $\mathcal{S}$ with respect to some metric $\mathcal{G}$ on $\mathcal{L}M$. The main aim of this subsection is to show that (generically, at least) $\mathcal{X}$ satisfies the conditions of §2, and thus construct the Morse complex $CM_*(\mathcal{L}M, L, \mathcal{G})$. $\mathcal{X}$ certainly admits a Lyapunov function, namely $\mathcal{S}$, and as $L$ is bounded below (by compactness of $M$) so is $\mathcal{S}$.

Unfortunately it is not necessarily the case that $\mathcal{X}$ is Morse; this depends on the choice of $V$. However the following theorem shows that it not too much of a restriction to suppose that $V$ is chosen such that $\mathcal{X}$ is Morse.

**Theorem 3.11.** *If $\mathcal{V} \subseteq C^\infty(S^1 \times M, \mathbb{R})$ denotes the set of potentials such that the associated Lagrangian action functional $\mathcal{S}$ is a Morse function*[8]*, then $\mathcal{V}$ is residual in $C^\infty(S^1 \times M, \mathbb{R})$. In other words, for generic $V$, $\mathcal{S}$ is Morse.*

The proof of Theorem 3.8 is a complicated transversality argument, which would unfortunately take too long to give here. The reader is referred to [W2], Theorem 1.1. From here on, we therefore suppose that $\mathcal{S}$ is Morse.

It is shown in [W2], Lemma 2.1 that $\mathcal{S}$ is a Morse function if and only if $\mathcal{A}$ is a Morse function. Moreover it is shown there that for any $V \in \mathcal{V}$ the following conditions are satisfied (actually the second condition follows from the first):

$$\det\left(\mathbb{1} - D\varphi^1(a(0))\right) \neq 0, \qquad \text{for all } a \in \mathcal{P}(H), \tag{3.26}$$

$$Z_c := \{a(0) \mid a \in \mathcal{P}(H), \mathcal{A}(a) \leq c\} \quad \text{is discrete for any } c \in \mathbb{R}. \tag{3.27}$$

We say such a Hamiltonian is **non-degenerate**.

The next thing to do is to check that all rest points have finite Morse indices.

**Theorem 3.12.** *Let $x \in \mathcal{P}(L)$; then the dimension of the largest subspace of $W^{2,2}(x^*TM)$ on which $D^2\mathcal{S}(x)(\cdot,\cdot)$ is negative definite is finite (where $W^{2,2}(x^*TM)$ denotes the $W^{2,2}$ sections of the pullback bundle $x^*TM$). Thus all the rest points of the vector field $\mathcal{X} = -\nabla\mathcal{S}$ have finite Morse index.*

[8] That is, none of its critical points are degenerate.

*Proof.* The messy bit of the proof involves calculating $D^2\mathcal{S}(x)$ for $x \in \mathcal{P}(L)$. One way to do this is via local coordinates, and a standard argument similar to the proof of Proposition 3.3 shows that for $\xi \in C^\infty(x^*TM)$ we have

$$D^2\mathcal{S}(x)(\xi,\xi) = \int_0^1 \left\langle -\nabla_{\dot{x}}\nabla_{\dot{x}}\xi - R(\xi,\dot{x})\dot{x} - \nabla_\xi \nabla V(t,x_s), \xi \right\rangle dt.$$

See for instance [J], §4.1, [K], Lemma 1.12.12, Lemma 2.5.1 or [W1] Lemma B.2.6. Moreover in the course of the proof one discovers that $D^2\mathcal{S}(x)(\cdot,\cdot)$ is a symmetric operator.

Define

$$A_x(\xi) := -\nabla_{\dot{x}}\nabla_{\dot{x}}\xi - R(\xi,\dot{x})(\dot{x}) - \nabla_\xi\nabla V(t,x).$$

We can consider $A_x$ as an unbounded operator in $L^2(S^1, x^*TM)$ with dense domain the space $W^{2,2}\left(S^1, x^*TM\right)$. The Morse index $m(x)$ of $x$ is the number of negative eigenvalues of $A_x$ counted with multiplicities. Observe that $A_x$ is self-adjoint, as it is made up of the self-adjoint operator $\frac{d^2}{dt^2}$ together with a bounded operator. Now use compactness of $M$ to obtain the existence of a constant $C > 0$ such that for all $\xi \in C^\infty(S^1, x^*TM)$.

$$\begin{aligned} \langle A_x(\xi),\xi\rangle_{L^2} &= \|\nabla_{\dot{x}}\xi\|_{L^2}^2 - \langle R(\xi,\dot{x})(\dot{x}),\xi\rangle_{L^2} - \left\langle \nabla_\xi\nabla V(t,x),\xi\right\rangle_{L^2} \\ &\geq \|\nabla_{\dot{x}}\xi\|_{L^2}^2 - C\|\xi\|_{L^2}^2. \end{aligned}$$

Thus if $\rho > C$ then the unbounded operator $A_x+\rho$ is positive definite and hence injective. Since $A_x+\rho$ is self-adjoint, it is also surjective with real spectrum, and viewing $A_x$ as a map $W^{2,2}(S^1, x^*TM) \to L^2(S^1, x^*TM)$, the Open Mapping Theorem gives us a bounded inverse

$$(A_x+\rho)^{-1} : L^2\left(S^1, x^*TM\right) \to W^{2,2}\left(S^1, x^*TM\right).$$

Composing with the compact embedding $W^{2,2}(S^1, x^*TM) \hookrightarrow L^2(S^1, x^*TM)$ (using the Sobolev Embedding Theorem), we discover that

$$(A_x+\rho)^{-1} : L^2\left(S^1, x^*TM\right) \to L^2\left(S^1, x^*TM\right)$$

is compact, and hence by the Spectral Theorem for self-adjoint compact operators has discrete spectrum $\{\lambda_i\}_{i\in\mathbb{N}}$, with finite multiplicities and only possible accumulation point being 0. Note that $\lambda$ is an eigenvalue of $(A_x+\rho)^{-1}$ if and only if $\lambda^{-1}-\rho$ is an eigenvalue of $A_x$. Then since each $\lambda_i > 0$, and $\lambda_i \to 0$, we see that $A_x$ has only finitely many negative eigenvalues. The proof is complete. ■

It remains to verify that the Palais-Smale condition and the Morse-Smale condition are satisfied. Under the assumptions we have made, the former is always satisfied, and we refer the reader to [Ben], Lemma 4.3 for a proof of this fact. The argument is complicated and fairly involved, although not technically too difficult.

The Morse-Smale condition is not always satisfied; this depends on the particular metric we use on $\mathcal{L}M$. Luckily, a generic metric works. More precisely, let $\Sigma$ denote the space of smooth sections of the bundle $\mathrm{End}(T\mathcal{L}M)$ such that for each $\sigma \in \Sigma$, $\sigma_x : T_x\mathcal{L}M \to T_x\mathcal{L}M$ is symmetric (with respect to the induced metric $\mathcal{G}_g$ on $\mathcal{L}M$) for all $x \in \mathcal{L}M$ and also such that $\|\sigma\|_{C^\infty} < \infty$ and $\|\sigma\|_{C^0} < 1$.

Given $\sigma \in \Sigma$, we obtain a complete Riemannian metric $\mathcal{G}_\sigma$ on $\mathcal{L}M$ defined by

$$\mathcal{G}_\sigma(\xi,\eta) := \mathcal{G}_g(\xi + \sigma_x(\xi),\eta), \qquad \xi,\eta \in T_x\mathcal{L}M,$$

which is uniformly equivalent to the original metric $\mathcal{G}_g$.

The following theorem is proved in [AM], Theorem 2.20. The argument is another complicated transversality argument.

**Theorem 3.13.** *There is a residual set $\Sigma' \subseteq \Sigma$ such that $\mathcal{G}_\sigma$ satisfies the Morse-Smale condition for every $\sigma \in \Sigma'$.*

Since every metric $\mathcal{G}_\sigma$ for $\sigma \in \Sigma$ is uniformly equivalent to the metric $\mathcal{G}_g$ on $\mathcal{L}M$ induced from our original metric $g$ on $M$, it follows that the Palais-Smale condition is still satisfied for any of the metrics $\mathcal{G}_\sigma$. The other conditions we were required to verify did not involve the metric. Hence if we choose a metric $\mathcal{G}_\sigma$ for some $\sigma \in \Sigma'$ then all of the required conditions are met (call such a metric a **Morse-Smale metric** for $\mathcal{S}$). Our work in §2 then immediately gives us:

**Theorem 3.14. (The isomorphism between $HM_*(\mathcal{L}M, L, \mathcal{G})$ and $H_*^{\mathrm{sing}}(\mathcal{L}M)$)**

*Let $L$ be a Lagrangian of the form* (3.11)*, where the potential $V$ is chosen such that the associated action functional $\mathcal{S}$ is Morse. Let $\mathcal{G}$ be a Morse-Smale metric on $\mathcal{L}M$, uniformly equivalent to the induced metric $\mathcal{G}_g$ on $\mathcal{L}M$ induced from the metric $g$ on $M$. Then we have a chain complex $CM_*(\mathcal{L}M, L, \mathcal{G})$ where*

*$CM_k(\mathcal{L}M, L, \mathcal{G})$ is the free abelian group generated by the critical points of $\mathcal{S}$ of Morse index $k$, and the boundary operator is given by* (2.3)*. Moreover the homology $HM_*(\mathcal{L}M, L, \mathcal{G})$ of this complex is isomorphic to the singular homology $H_*^{sing}(\mathcal{L}M)$ of $\mathcal{L}M$.*

*Moreover exactly the same result holds if instead our Lagrangian $L^{\text{tw}}$ of the form* (3.24) *is used.*

*Remark.* The last statement of the Theorem is not immediate. Our proof of Theorem 3.12 explicitly used the fact that $L$ was of the form (3.11); this however can easily be fixed. Both Theorem 3.11 and Theorem 3.13 hold in sufficient generality that they can cope with $L^{\text{tw}}$; see [W2] and [AM] respectively.

Before finishing this subsection, we note a crucial result that we will need later on when we set up the Floer complex.

**Lemma 3.15.** *For every $c \in \mathbb{R}$, the set of solutions $a \in \mathcal{P}(H)$ satisfying $\mathcal{A}(a) \le c$ is finite.*

*Proof.* Let $a(t) = (x(t), p(t)) : S^1 \to T^*M$ be a critical point of $\mathcal{A}$ such that $\mathcal{A}(a) \le c$. Then by (3.10) and (3.5), we have

$$\begin{aligned} c \ge \mathcal{A}(a) &= \int_0^1 p(t)\,(\dot{x}(t)) - H(t, x(t), p(t))dt \\ &= \int_0^1 \dot{x}(t)^\flat(\dot{x}(t)) - \frac{1}{2}|p(t)|^2 - V(t, x(t))dt \\ &= \int_0^1 \frac{1}{2}\,|p(t)|^2 - V(t, x(t))dt \\ &\ge \frac{1}{2}\,\|p\|_{L^2}^2 + C, \end{aligned}$$

for a constant $C$. Thus $\mathcal{P}(H) \cap \{\mathcal{A} \le c\}$ is bounded in $L^2$. Next, since

$$|\dot{a}(t)| = |X_H\,(t, a(t))\,|,$$

and

$$X(t, x(t), p(t)) \doteqdot \left(p(t)^\#, -dV_{(t,x(t))}\right),$$

by (3.20), it follows that $\mathcal{P}(H) \cap \{\mathcal{A} \le c\}$ is bounded in $W^{1,1}$, and hence also in $L^\infty$. In particular, the set $Z_c := \{a(0) | a \in \mathcal{P}(H), \mathcal{A}(a) \le c\}$ is precompact in $T^*M$, and then the non-degeneracy condition (3.27) implies $A$ is finite. ∎

## 4. Floer Homology

Throughout this section, we shall suppose $H$ is a Hamiltonian of the form (3.10), where the potential $V$ is an element of $\mathcal{V}$, and so the non-degeneracy conditions (3.26) and (3.27) are satisfied.

### The Conley-Zehnder index

The aim of this subsection is to obtain a unique integer $\mu(a)$ for each $a \in \mathcal{P}(H)$. This index will play the role of the Morse index $m(x)$ for $x \in \mathcal{P}(L)$; Theorem 4.2 below shows exactly how the two concepts are related.

Before defining the integer $\mu(a)$ we need to first introduce the related concept of the Conley-Zehnder index $\mu_{CZ}$. Let $\text{Sp}(2n)$ denote the set of $2n \times 2n$ symplectic matrices, and $\text{Sp}^*(2n)$ the open dense set of $\text{Sp}(2n)$ given by

$$\text{Sp}^*(2n) := \{A \in \text{Sp}(2n) \mid \det(\mathbb{1} - A) \neq 0\}.$$

Now let $\mathscr{S}(2n)$ denote the set of continuous paths $h : [0, 1] \to \text{Sp}(2n)$ such that $h(0) = \mathbb{1}$ and $h(1) \in \text{Sp}^*(2n)$.

**Definition 4.1.** The **Conley-Zehnder index** $\mu_{CZ}$ assigns an integer $\mu_{CZ}(h)$ to each path $h \in \mathscr{S}(2n)$. It satisfies the following two properties:

(1) **(Homotopy***)* $\mu_{CZ}$ is constant on the components of $\mathscr{S}(2n)$; if $h$ and $h'$ are elements of $\mathscr{S}(2n)$ that are homotopic by a homotopy that fixes the end points then $\mu_{CZ}(h) = \mu_{CZ}(h')$,

(2) **(Naturality)** if $h \in \mathscr{S}(2n)$ and $f : [0, 1] \to \text{Sp}(2n)$ is any continuous path then

$$\mu_{CZ}(h) = \mu_{CZ}(fhf^{-1}).$$

There are several equivalent ways to define the Conley-Zehnder index (see for instance [Sa], §2.4 or [SZ]). Here is one simple one.

It is shown in [SZ] that there exists a unique continuous map $\rho : \mathrm{Sp}(2n) \to S^1$ satisfying the following properties:

- $\rho|_{U(n)} = \det : U(n) \to S^1$ (recall that $U(n) = \mathrm{Sp}(2n) \cap O(n)$),
- if $A \in \mathrm{Sp}(2n)$ and $B \in \mathrm{Sp}(2m)$ (so $A \oplus B \in \mathrm{Sp}(2n+2m)$) then $\rho(A \oplus B) = \rho(A)\rho(B)$,
- $\rho(BAB^{-1}) = \rho(A)$,
- $\rho(A) = \pm 1$ if $A$ has no eigenvalues on $S^1$.

If $h \in \mathscr{S}(2n)$ then one can show that there exists a unique (up to homotopy) continuous path $\tilde{h} : [0,2] \to \mathrm{Sp}(2n)$ such that $\tilde{h}|_{[0,1]} = h$, $\tilde{h}(s) \in \mathrm{Sp}^*(2n)$ for all $s \geq 1$ and $\tilde{h}(2) = -\mathbb{1}$ or the matrix $W(2n)$ where

$$W(2n) := \begin{pmatrix} 2 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & \frac{1}{2} & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix} \quad \text{(illustrated for } n = 3).$$

Then it is easily seen that $\rho^2 \circ \tilde{h} : [0,2] \to S^1$ is a loop and thus has a well defined degree; we define

$$\mu_{CZ}(h) := \deg\left(\rho^2 \circ \tilde{h}\right).$$

We now wish to associate an index $\mu(a)$ for any $a \in \mathcal{P}(H)$. Given a critical point $a : S^1 \to T^*M$ choose a map $r : D^2 \to T^*M$ such that $r(e^{2\pi it}) = a(t)$; such a map exists as $T^*M$ is simply connected, since $M$ is[9]. Since $D^2$ is contractible, the pullback bundle $r^*(TT^*M) \to D^2$ is contractible, and a trivialisation of it determines a **unitary trivialisation** $\Phi : S^1 \times \mathbb{R}^{2n} \to a^*(TT^*M)$ of the pullback bundle $a^*(TT^*M)$.

More specifically, there exists a trivialisation

$$D^2 \times \mathbb{R}^{2n} \to r^*(TT^*M), \qquad (z, v) \mapsto \Phi(z)v,$$

such that $J\Phi = \Phi J_0$, $\Phi^*\omega = \omega_0$ and $\Phi^* g_J = g_0$, where $g_0$ denotes the Euclidean scalar product on $\mathbb{R}^n$. See [SZ], Lemma 5.1 for a proof of the existence of such a $\Phi$. Moreover, we claim any two such trivialisations are homotopic: indeed if $\Phi'$ is another such trivialisation then $z \mapsto \Phi'(z)^{-1}\Phi(z)$ is a smooth map $D^2 \to U(n)$, which is therefore homotopic to the constant map $z \mapsto 1$. Hence given $r$ we have a well defined unitary trivialisation of the pullback bundle $a^*(TT^*M)$, given by restriciting $\Phi$ to $S^1$. Of course for this to be meaningful we also want the choice of $\Phi$ to be independent of our original choice of $r$. In order for this to be so, it is enough to be able to show that the first Chern class $c_1(TT^*M)$ vanishes over $\pi_2(T^*M)$, see [SZ], Lemma 5.2. In fact, $c_1(TT^*M) = 0$; an elementary proof of this is given in [W1], Theorem B.1.9, and so this is no problem.

Now define the path $h : I \to \mathrm{Sp}(2n)$ by

$$h(t) := \Phi(t)^{-1} \circ D\varphi^t_{a(0)} \circ \Phi(0).$$

Since $\Phi$ is unitary, $h(t) \in U(n) \subseteq \mathrm{Sp}(2n)$ for all $t$. Certainly $h_a(0) = \mathbb{1}$, and the non-degeneracy condition (3.26) implies that $h_a(1) \in \mathrm{Sp}^*(2n)$. Thus $h \in \mathscr{S}(2n)$, and we define the **index** $\mu(a)$ of $a$ by

$$\mu(a) := \mu_{CZ}(h_a).$$

It follows from the naturality and homotopy properties of $\mu_{CZ}$ that this is well defined; see [Sa] §2.6 for the details.

We conclude this subsection with the following theorem, apparently proved first by Duistermaat, and generalised by Weber in [W2], Theorem 1.2, where we refer the reader for the proof. It will prove absolutely crucial in §5 when we come to prove the isomorphism between the Morse and Floer complexes.

**Theorem 4.2. (The Index Theorem)**

*Let $a \in \mathcal{P}(H)$, and let $x \in \mathcal{P}(L)$ denote the solution corresponding to $a$ (so $\dot{a} \doteqdot (\dot{x}, \dot{x}^\flat)$). Then*

$$\mu(a) = m(x).$$

[9]Here is one case where making the assumption that $M$ is simply connected simplifies things. Without this assumption more work is need here; see [AS], §1.2.

**Floer's equation and the trajectory spaces**

We now begin the constuction of the Floer complex. As mentioned before, we cannot simply proceed as in Morse case, as the action functional $\mathcal{A}$ does not satisfy the required conditions; its critical points need not have finite Morse index, for instance. Floer's brilliant idea was to study the gradient equation as a certain elliptic PDE:

**Definition 4.3.** Fix a smooth time dependent family of $\omega$-compatible almost complex structures $J : S^1 \times T^*M \to T^*M$, and write $J_t = J(t, \cdot)$. Note that the compatibility condition allows us to rewrite (3.14) as

$$d\mathcal{A}_a(\xi) = \int_0^1 \langle \xi, -J_t(a)\,(\dot{a} - X(t, \dot{x}))\rangle_{J_t}\, dt. \tag{4.1}$$

The PDE we wish to study is the **Floer's negative gradient equation**: given $a, b \in \mathcal{P}(H)$, we look for solutions $u \in C^\infty(\mathbb{R} \times S^1, T^*M)$ satisfying the nonlinear **perturbed Cauchy-Riemann** operator $\bar{\partial}_{H,J}$:

$$\begin{aligned} \bar{\partial}_{H,J}(u) := \partial_s u - J_t(u)\,(\partial_t u - X(t, u)) &= 0 \\ = \partial_s u - J_t(u)\partial_t u + \nabla H(t, u) &= 0, \end{aligned} \tag{4.2}$$

$$\lim_{s\to\infty} u(s, t) = a(t),\ \lim_{s\to-\infty} u(s, t) = b(t) \text{ uniformly in } t. \tag{4.3}$$

We let $\mathcal{T}(a, b) = \mathcal{T}(a, b, H, J)$ denote the set of solutions to (4.2) and (4.3). The spaces $\mathcal{T}(a, b)$ are called the **trajectory spaces** (later on we will introduce the **moduli spaces** $\mathcal{M}(a, b) = \mathcal{T}(a, b)/\mathbb{R}$) . We wish to exhibit the spaces $\mathcal{T}(a, b)$ as smooth finite dimensional submanifolds of an appropriate Banach manifolds of curves.

We assume that the reader is familiar with the concept of a Fredholm operator between Banach spaces, and what it means for a smooth map between Banach manifolds to be Fredholm. A good reference for this is §A.3 of [MS3]. A related concept is a **semi-Fredholm** operator, which is an operator with closed range and finite dimensional kernel. The following lemma will prove useful, whose proof can be found in [Sc1], Lemma 2.13.

**Lemma 4.4.** *Let $X, Y$ and $Z$ be Banach spaces, $F : X \to Y$ continuous, $K : X \to Z$ compact. Suppose there exists $c > 0$ such that for all $x \in X$,*

$$\|x\|_X \le c\,(\|Fx\|_Y + \|Kx\|_Z)\,.$$

*Then $F$ is a semi-Fredholm operator.*

We will consider the operator $\bar{\partial}_{H,J}$ as an operator

$$u \mapsto \bar{\partial}_{H,J}(u) \in C^\infty\left(u^*\left(TT^*M\right)\right)$$

on the space

$$\left\{u \in C^\infty(\mathbb{R} \times S^1, T^*M) \,\Big|\, \lim_{s\to\infty} u(s, t) = a(t),\ \lim_{s\to-\infty} u(s, t) = b(t)\right\}, \tag{4.4}$$

and so the trajectory space $\mathcal{T}(a, b)$ is the preimage $\bar{\partial}_{H,J}^{-1}(0)$. The first step is to complete (4.4) into a Banach manifold $\mathcal{B}(a, b)$ and extend $\bar{\partial}_{H,J}$ as a smooth mapping. It turns out that the correct way to do this to complete (4.4) with respect the Sobolev topology of $W^{1,r}\left(\mathbb{R} \times S^1, T^*M\right)$ $(r > 2)$. We thus obtain a Banach manifold

$$\mathcal{B}(a, b) \subseteq \left\{u \in C^0(\mathbb{R} \times S^1, T^*M) \,\Big|\, \lim_{s\to\infty} u(s, t) = a(t),\ \lim_{s\to-\infty} u(s, t) = b(t), \text{ uniformly in } t\right\}.$$

In fact, the actual construction is slightly more involved; to compensate for the fact that (unlike in the standard closed case) $T^*M$ is non-compact we need to impose additional asymptotic conditions on elements of $\mathcal{B}(a, b)$. The precise definition is:

**Definition 4.5.** Fix some $r > 2$. Define $\mathcal{B}(a, b)$ to be the space of maps $u \in W^{1,r}_{\mathrm{loc}}\left(\mathbb{R} \times S^1, T^*M\right)$ such that there exists $s_0 \in \mathbb{R}$ and $W^{1,r}$ sections $\alpha, \beta$ of the bundles $a^*\left(TT^*M\right)$ and $b^*\left(TT^*M\right)$ such that

$$u(s, t) = \begin{cases} \exp_{a(t)}\left(\alpha(s, t)\right) & s \le -s_0 \\ \exp_{b(t)}\left(\beta(s, t)\right) & s \ge s_0, \end{cases}$$

where we view $a^*\left(TT^*M\right)$ and $b^*\left(TT^*M\right)$ as bundles over $(-\infty, -s_0] \times S^1$ and $[s_0, \infty) \times S^1$.

Here is the explanation for the condition $r > 2$. Recall from the Morrey's Inequality (see [E],Chapter 5, Theorem 4) that for $\mathbb{R}^2$ and for $r > 2$ we have a continuous embedding $W^{1,r}(\mathbb{R}^2) \hookrightarrow C^{0,1-2/r}(\mathbb{R}^2)$, that is, up to a set of measure zero, $u \in W^{1,r}(\mathbb{R}^2)$ is actually Hölder continuous, but there exist $W^{1,2}$ functions on $\mathbb{R}^2$ that are discontinuous. The proof of Theorem 4.7 will implicitly require (at least) continuity.

$\mathcal{B}(a,b)$ is modelled on $W^{1,2}\left(\mathbb{R} \times S^1, T^*M\right)$; given $u \in \mathcal{B}(a,b)$ the tangent space $T_u\mathcal{B}(a,b)$ is identified with $W^{1,r}(u^*(TT^*M))$. There is a Banach bundle $\hat{\mathcal{B}}(a,b) \to \mathcal{B}(a,b)$ whose fibre over $u \in \mathcal{B}(a,b)$ is the space of $L^r$ sections of $u^*(TT^*M)$:

$$\hat{\mathcal{B}}(a,b) = \coprod_{u \in \mathcal{B}(a,b)} \{u\} \times L^r(u^*(TT^*M)).$$

The reader is referred to [Sc1], Appendix A for a detailed explanation of the methodology behind the construction of $\mathcal{B}(a,b)$ and $\hat{\mathcal{B}}(a,b)$. We have the following result:

**Proposition 4.6.** *We may extend $\bar{\partial}_{H,J}$ to obtain a smooth section*

$$\bar{\partial}_{H,J} : \mathcal{B}(a,b) \to \hat{\mathcal{B}}(a,b)$$

*such that $\bar{\partial}_{H,J}^{-1}(0) = \mathcal{T}(a,b)$.*

The main thing to prove above is that solutions $u \in \bar{\partial}_{H,J}^{-1}(0)$ are in fact smooth: this follows from elliptic regularity.

A much more difficult result is:

**Theorem 4.7. (The Fredholm property of $\bar{\partial}_{H,J}$)**
*The linearisation $D_u = \left(D\bar{\partial}_{H,J}\right)_u : W^{1,r}(u^*(TT^*M)) \to L^r(u^*(TT^*M))$ is a Fredholm operator.*

*Proof. (sketch)*
Since the boundary loop $a$ is contractible, the symplectic vector bundle $u^*(TT^*M)$ is trivial, and we may obtain a unitary trivialisation $\Phi : \left(\mathbb{R} \times S^1\right) \times \mathbb{R}^{2n} \to u^*(TT^*M)$ (see the discussion of the Conley-Zehnder index above). The nondegeneracy assumption (3.26) on $H$ has the important consequence that solutions $u \in \bar{\partial}_{H,J}^{-1}(0)$ converge exponentially fast with their derivatives as $|s| \to \infty$, that is, $u(s,t) \to a(t)$ (resp. $b(t)$) exponentially fast as $s \to \pm\infty$ and $\partial_s(u(s,t)) \to 0$ as $|s| \to \infty$ exponentially fast; see [Sa], Lemma 2.11. This allows us to assume that $\Phi$ is smoothly extended over an appropriate compactification of the cylinder $\bar{\mathbb{R}} \times S^1$ (where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$).

The linearised operator $D_u$ conjugated by such a trivialisation is of the form

$$D := \Phi^{-1} \circ D_u \circ \Phi : W^{1,r}\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right) \to L^r\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right),$$

$$D = \partial_s + J_0\partial_t + S, \qquad S \in C^\infty(\bar{\mathbb{R}} \times S^1, \mathrm{Mat}_{2n}(\mathbb{R}))$$

where $S(s,t)$ is a smooth matrix valued map on the compactified cylinder. Letting

$$T(s) := J_0\partial_t + S(s,\cdot) : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2\left(S^1, \mathbb{R}^{2n}\right),$$

the non-degeneracy of $a$ and $b$ imply that $T(\pm\infty)$ are self adjoint injective operators, and thus isomorphisms. Hence the linearisation $D = \partial_s + T(s)$ may be viewed as a 1-parameter family of operators which are asymptotically self adjoint isomorphisms.

We want to show $D$ is Fredholm, and we will begin by showing that $D$ is semi-Fredholm. The key step is the following inequality.

*There exists $C, c > 0$ such that for all $\sigma \in W^{1,r}\left(\mathbb{R} \times S^1\right)$,*

$$\|\sigma\|_{W^{1,r}(\mathbb{R}\times S^1)} \le C\left(\|D\sigma\|_{L^r(\mathbb{R}\times S^1)} + \left\|\sigma|_{[-c,c]\times S^1}\right\|_{L^r([-c,c]\times S^1)}\right). \tag{4.5}$$

This, together with the previous lemma imply the semi Fredholm property, since we have a compact embedding

$$W^{1,r}\left(\mathbb{R} \times S^1\right) \hookrightarrow L^r\left([-c,c] \times S^1\right), \qquad \sigma \mapsto \sigma|_{[-c,c]\times S^1}.$$

The proof of (4.5) is difficult, and requires two different inequalities. The first is a standard elliptic estimate for the standard Cauchy-Riemann operator

*Let $\Omega \subseteq \mathbb{R}^2$ be an open domain. Choose $1 < r < \infty$ and $k \in \mathbb{N}$. Then every weak solution $u \in L^r_{\mathrm{loc}}(\Omega)$ of $\bar{\partial}(u) = f$ for $f \in W^{k,r}_{\mathrm{loc}}(\Omega)$ satisfies $u \in W^{k+1,r}_{\mathrm{loc}}(\Omega)$. Moreover for any subdomain $\Omega' \subset\subset \Omega$ there exists a constant $C = C(k, r, \Omega', \Omega)$ such that for all $u \in C^\infty\left(\overline{\Omega}\right)$ it holds that*

$$\|u\|_{W^{k+1,r}(\Omega')} \leq C\left(\left\|\bar{\partial}(u)\right\|_{W^{k,r}(\Omega)} + \|u\|_{L^r(\Omega)}\right).$$

This statement is proved in [MS1], Theorem $B.3.4$. From this one deduces the following result, based on [Sc2], Corollary 2.5.3.

*If $u \in L^r_{\mathrm{loc}}(\mathbb{R} \times S^1)$ is a weak solution of $Du = g$ with $g \in W^{k,r}_{\mathrm{loc}}(\mathbb{R} \times S^1)$ then $u \in W^{k+1,r}_{\mathrm{loc}}(\mathbb{R} \times S^1)$. Moreover for any subdomain $\Omega \subset\subset \mathbb{R} \times S^1$ there exists a constant $C = C(k, r, S, \Omega)$ such that*

$$\|u\|_{W^{k+1,r}(\Omega)} \leq C\left(\|Du\|_{W^{k,r}(\mathbb{R}\times S^1)} + \|u\|_{L^r(\mathbb{R}\times S^1)}\right). \tag{4.6}$$

This is simple to deduce from the previous statement, as $u$ is a weak solution of $Du = g$ if and only if $u$ is a weak solution of $\bar{\partial}(u) = f$, where $f = g - Su \in L^r_{\mathrm{loc}}(\mathbb{R} \times S^1)$. Thus by induction on $k$ we obtain $u \in W^{k,r}_{\mathrm{loc}}(\mathbb{R} \times S^1)$, and moreover

$$\begin{aligned} \|u\|_{W^{k+1,r}(\Omega)} &\leq C\left(\left\|\bar{\partial}(u) + Su - Su\right\|_{W^{k,r}(\mathbb{R}\times S^1)} + \|u\|_{L^r(\mathbb{R}\times S^1)}\right) \\ &\leq C\left(\|Du\|_{W^{k,r}(\mathbb{R}\times S^1)} + \|Su\|_{W^{k,r}(\mathbb{R}\times S^1)} + \|u\|_{L^r(\mathbb{R}\times S^1)}\right), \end{aligned}$$

for $C = C(k, r, \Omega', \Omega)$, and then using the fact that $S$ is smooth we obtain the stated estimate, with $C$ now depending additionally on $S$.

This however is not enough; a further estimate related to the boundary condition is also required. Letting $D_\pm := \partial_s + T(\pm\infty)$, in [Sc2], Theorem 3.1.3 it is proved that there exist constants $C_\pm$ such that for all $\sigma \in W^{1,r}\left(\mathbb{R} \times S^1\right)$,

$$\|\sigma\|_{W^{1,r}\left(\mathbb{R}\times S^1\right)} \leq C_\pm \|D_\pm u\|_{L^r(\mathbb{R}\times S^1)}. \tag{4.7}$$

This implies that the asymptotic translation invariant operators $D_\pm$ are isomorphisms. The two inequalities (4.6) and (4.7) then together imply (4.5); see [Sc2], p54 for all the details.

Having proved (4.5), the proof follows readily. Indeed, we consider the (formal) adjoint

$$D^* = -\partial_s + J_0\partial_t + S(s,t)^t.$$

This is of the same form as $D$, after replacing $J_0$ by $-J_0$. Thus arguing as before one discovers that $D^*$ is also semi-Fredholm. It remains to relate the cokernel of $D$ to the kernel of $D^*$, which is done using the regularity results for weak solutions combined with the asymptotic decrease of weak solutions of $D^*$. We see that $\ker D^* = \mathrm{coker} D$ (see [Sc2], Proposition 3.1.30) and this completes the proof. of the Fredholm property. ∎

The next thing to do is to compute the index of the operator $D_u$. One possible way to do this is to study the **spectral flow** of the operator $D_u$: the spectral flow $\mu_{\mathrm{spec}}(D_u)$ is shown to equal both $\mathrm{ind} D_u$ as well as $\mu(a) - \mu(b)$. For reasons of space we won't discuss the spectral flow, and thus refer the reader to [Sa], §2.5 for this part of the story; complete proofs of the statements referred to there can be found in [RS] §4, which also gives a comprehensive exposition of all the topics referred to here. We conclude that $\bar{\partial}_{H,J}$ is Fredholm of index $\mu(a) - \mu(b)$.

If we knew that 0 was a regular value of $\bar{\partial}_{H,J}$ then we could apply a suitable (infinite dimensional version of the) Inverse Function Theorem (see Theorem A.3.3 in [MS3] to conclude that $\mathcal{B}(a,b) = \bar{\partial}^{-1}_{H,J}(0)$ is a submanifold of dimension $\mu(a) - \mu(b)$. Unfortunately, 0 is not necessarily a regular value of $\bar{\partial}_{H,J}$; this is a condition that depends on the time dependent almost complex structure $J$. This leads us to define:

**Definition 4.8.** Let $\mathcal{J}$ denote the set of all time dependent complex structures such that $J_t$ is compatible with $\omega$ and $\left\|J_t - J_g\right\|_\infty < \infty$ for all $t$. Let $\mathcal{J}_{\mathrm{reg}}(H) \subseteq \mathcal{J}$ denote the set of all time dependent almost complex structures $J \in \mathcal{J}$ such that $\bar{\partial}_{H,J} : \mathcal{B}(a,b) \to \hat{\mathcal{B}}(a,b)$ is transverse to the zero section for all $a, b \in \mathcal{P}(H)$, that is 0 is a regular value of $\bar{\partial}_{H,J} : \mathcal{B}(a,b) \to \hat{\mathcal{B}}(a,b)$ for all $a, b \in \mathcal{P}(H)$. We call such almost complex structures **regular** (with respect to $H$).

$\mathcal{J}$ can be made into a complete metric space, and in [FHS], Theorem 5.1, the following result is proved[10]:

[10] Actually in [FHS] the result is proved for a compact symplectic manifold $(M, \omega)$ ; since $T^*M$ is non-compact, this result doesn't directly apply. However (cf. Remark 5.4 of [FHS]) by restricting the growth of the Hamiltonian $H$ the result does still hold - and the assumption that $H$ is of the form (3.10) is sufficient for this.

**Theorem 4.9.** *Let $H$ denote a Hamiltonian of the form* (3.10)*. The set $\mathcal{J}_{reg}(H)$ is residual in $\mathcal{J}$.*

We summarize the results of this subsection with the following theorem.

**Theorem 4.10. (The Trajectory Space Theorem)**
*Let $H$ be a non-degenerate Hamiltonian of the form* (3.10)*, and let $J \in \mathcal{J}_{reg}(H)$. Then the trajectory spaces $\mathcal{T}(a, b, H, J)$ are all smooth manifolds of dimension $\mu(a) - \mu(b)$.*

**Precompactness**

In 'standard' Floer theory, that is, Floer theory on a closed symplectic manifold, the next step is to show that the trajectory spaces $\mathcal{T}(a, b)$ are all $C^\infty_{\text{loc}}$ precompact. Unfortunately since $T^*M$ is non-compact, in our case this presents a significant problem. This is where a crucial result proved by Abbondandolo and Schwarz ([AS], Theorem 1.9.1) enters. In what follows, given $u(s, t) \in C^\infty(\mathbb{R} \times S^1, T^*M)$, we let $u_s(t) : S^1 \to T^*M$ denote the loop on $T^*M$ given by $u_s(t) := u(s, t)$.
The precise statement is as follows:

**Theorem 4.11.** **($L^\infty$ gradient estimates)**
*There exists $j_0 > 0$ such that if $J$ is a time dependent 1-periodic almost complex structure on $T^*M$ satisfying $\|J_t - J_g\|_\infty < j_0$ for all $t$ then for every $c, d \in \mathbb{R}$, the set of solutions $u \in C^\infty(\mathbb{R}\times S^1, T^*M)$ of* (4.2) *satisfying $c \le \mathcal{A}(u_s) \le d$ for all $s \in \mathbb{R}$ is bounded in $L^\infty(\mathbb{R} \times S^1, T^*M)$.*

We shall not go into the proof of Theorem 4.11; the proof is long, intricate and technical. As mentioned above, the reader is referred to [AS], Theorem 1.9.1 for the details. Instead let us show how the $L^\infty$ estimates allow us to obtain the desired compactness results. Let $\mathcal{J}_{j_0,\text{reg}}(H)$ denote the subset of $\mathcal{J}_{\text{reg}}(H)$ consisting of those almost complex structures satisfying $\|J_t - J_g\|_\infty < j_0$ for all $t$.
The result we have in mind is the following, which the reader is invited to compare to the previous precompactness result proved in Corollary 2.9:

**Theorem 4.12. (The Precompactness Theorem for $\mathcal{T}(a, b)$)**
*Let $J \in \mathcal{J}_{j_0,reg}(H)$. Then the trajectory spaces $\mathcal{T}(a, b, H, J)$ are precompact in $C^\infty_{loc}(\mathbb{R} \times S^1, T^*M)$ for all $a, b \in \mathcal{P}(H)$.*

*Remark.* If instead of $T^*M$ we worked with a compact symplectic manifold $N$ then this would be almost immediate: if $K \subseteq \mathbb{R} \times S^1$ is a compact subset and $(u_n) \in \mathcal{T}(a, b)$ then compactness of $N$ would give uniform bounds on the $(u_n|_K)$ and all their derivatives, whence the Arzelà-Ascoli theorem would yield some $u : K \to N$ and a subsequence $(n_j)$ such that $u_{n_j}|_K \xrightarrow{C^\infty} u$. Taking $K_n = [-n, n] \times S^1$ and letting $n \to \infty$ we conclude that if $(u_n) \in \mathcal{T}(a, b)$ then $(u_n)$ admits a subsequence $(u_{n_j})$ converging in the $C^\infty_{\text{loc}}$-topology to some $u \in \overline{\mathcal{T}(a, b)}$. Note also that by continuity we have $\bar\partial_{H,J}(u) = 0$.

Once we have proved the Precompactness Theorem we will have enough information at our disposal to construct the Floer complex $CF_*(T^*M, \omega, H, J)$.
We begin our proof of Theorem 4.12 with a definition.

**Definition 4.13.** Let $(N, J)$ and $(N', J')$ be almost complex manifolds (that is, manifolds admitting an almost complex structure). A smooth map $\varphi : N \to N'$ is called **$(J, J')$-holomorphic** if the differential $D\varphi_x : T_xN \to T_{\varphi(x)}N'$ is complex linear for all $x \in N$, that is, if and only if for all $x \in N$,

$$D\varphi_x \circ J_x = J'_{\varphi(x)} \circ D\varphi_x.$$

In particular if $v : \mathbb{C} \to N$ is smooth, we say that $v$ is a **$J$-holomorphic curve** if $v$ is $(i, J)$-holomorphic, where $i$ is the standard almost complex structure on $\mathbb{C}$. If we write the coordinates on $\mathbb{C}$ as $s + it$, this amounts to $v$ solving the Cauchy-Riemann equation

$$\bar\partial_J(v) = \partial_s v + J(v)\partial_t v = 0.$$

Similarly if we let $j$ denote the standard almost complex structure on $S^2$ we say that a smooth $(j, J)$-holomorphic map $v : S^2 \to N$ is a **$J$-holomorphic sphere.**

There are no non-constant $J$-holomorphic spheres on $T^*M$:

**Lemma 4.14.** *Let $J$ be an $\omega$-compatible almost complex structure on $T^*M$. Then the only $J$-holomorphic spheres on $T^*M$ are constants.*

*Proof.* The trick is to consider the **energy** $E(v)$ of $v$, defined to be the $L^2$ norm of the 1-form $dv \in \Omega^1(v^*(TT^*M))$,

$$E(v) := \frac{1}{2}\int_{S^2} |dv|_J^2,$$

where $|\cdot|_J$ is the norm induced by the metric $\langle\cdot,\cdot\rangle_J$. Note that $E(v) > 0$ if $v$ is non-constant. Now one observes that

$$E(v) = \int_{S^2} v^*(\omega).$$

If we write the coordinates on $S^2$ as $s+it$, we have

$$\begin{aligned}\frac{1}{2}|dv|_J^2 &= \frac{1}{2}\left(|\partial_s v|_J^2 + |\partial_t v|_J^2\right) ds\wedge dt \\ &= \frac{1}{2}|\partial_s v + J(v)\partial_t v|_J^2\, ds\wedge dt - \langle\partial_s v, J(v)\partial_t v\rangle_J\, ds\wedge dt \\ &= \left|\bar\partial_J(v)\right|_J^2 dS + \frac{1}{2}\left(\omega(\partial_s v,\partial_t v) + \omega(J(v)\partial_s v, J(v)\partial_t v)\right) ds\wedge dt \\ &= 0 + v^*(\omega),\end{aligned}$$

where on the penultimate line we used $\omega(v,w) = -\langle v, Jw\rangle_J$ and $\omega(Jv,Jw) = \omega(v,w)$. Given this, we conclude by observing by Stokes' Theorem that

$$E(v) = \int_{S^2} v^*(\omega) = \int_{S^2} v^*(d\theta) = \int_{S^2} d(v^*(\theta)) = 0,$$

■

We now quote the following result from [AL].

**Theorem 4.15. (The removable singularity theorem for cotangent bundles)**
*Let $J$ be compatible almost complex structure on $T^*M$, and let $v:\mathbb{C}\to T^*M$ be a $J$-holomorphic curve with finite energy, Then $v$ can be extended to a $J$-holomorphic sphere $\tilde v: S^2\to T^*M$.*

*Remark.* This theorem is often stated for compact manifolds[11] only: that is, if $(N,J)$ is a compact symplectic manifold and $v:\mathbb{C}\to N$ is a $J$-holomorphic curve with finite energy then $v$ extends to a $J$-holomorphic sphere (see for instance [Sa], Remark 4.1 or [MS3], Theorem 4.1.2). However a stronger result is true; in [AL], Chapter V, Theorem 4.5.1 the same result is proved for all **tame** symplectic manifolds; a much wider class of symplectic manifold that includes the cotangent bundles (see §4.2 of that chapter).

The next result we quote concerns convergence of a sequence of $J$-holomorphic curves.

**Theorem 4.16.** *Let $D_k \subseteq D_{k+1} \subseteq \mathbb{R}^2$ be discs about $0$ such that $\mathbb{R}^2 = \bigcup_{k\in\mathbb{N}} D_k$, and suppose $v_k : D_k \to T^*M$ are $J$-holomorphic curves such that*

$$|\nabla v_k(x)|_J \le 2 \text{ for all } x\in D_k,$$

$$|\nabla v_k(0)|_J = 1,$$

$$\int_{D_k} |\nabla v_k|_J^2 \le C \text{ for some constant } C\ge 0 \text{ and for all } k\in\mathbb{N}.$$

*Then there exists a subsequence $(k_j)$ and a $J$-holomorphic curve $v:\mathbb{R}^2\to T^*M$ such that $v_{k_j}\to v$ in the $C^\infty_{\text{loc}}$-topology.*

For reasons of space we won't prove this result, although it is not technically very difficult; the proof can be found in [HZ] Chapter 6, Lemma 6. Note that whilst they work with a compact manifold only throughout, in this lemma they do not actually use compactness. A more general result (which does not use compactness) can be found in Theorem B.4.2 in [MS1].

The key auxiliary proposition that will allow us to deduce the Precompactness Theorem is the following.

**Proposition 4.17.** *Fix $a,b\in\mathcal{P}(H)$. There exists $C\ge 0$ such that for all $u\in\mathcal{T}(a,b)$ and all $(s,t)\in\mathbb{R}\times S^1$, $|\nabla u(s,t)|_J\le C$.*

[11] Which confused me for a quite a long time.

*Proof.* We will show some that if the result is false then by judicious rescaling we may find sequence such that the hypotheses of Theorem 4.16 are satified. Then we use the Removable Singularity Theorem 4.15 to obtain a non-constant $J$-holomorphic sphere, which then contradicts Lemma 4.14. Here are the details (in order to simplify the notation we write $|\cdot|$ instead of $|\cdot|_J$ throughout this proof):

For this proof, we shall consider the gradient lines $u : \mathbb{R} \times S^1 \to T^*M$ as being defined on $\mathbb{R}^2$ (so $u(s, t+1) = u(s,t)$ for all $t$). If the results fails, we can find $u_k \in \mathcal{T}(a,b)$ and $(s_k, t_k) \in \mathbb{R}^2$ such that $|\nabla u_k(s_k, t_k)| \to \infty$. In fact, we can do more: we can choose $\epsilon_k \to 0$ such that

$$\epsilon_k \, |\nabla u_k(s_k, t_k)| \to \infty.$$

A simple argument in elementary point-set topology (given in [HZ], Chapter 6, Lemma 5) allows us to assume in addition that for all $(s,t) \in B_{\epsilon_k}((s_k, t_k))$ we have

$$2 \, |\nabla u_k(s_k, t_k)| \geq |\nabla u_k(s,t)| \, .$$

Now set $N_k := |\nabla u_k(s_k, t_k)|$, and let

$$v_k(s,t) := u_k\left(s_k + \frac{s}{N_k}, t_k + \frac{t}{N_k}\right).$$

Then $|v_k(0,0)| = 1$, and if $D_k$ denotes the disk $B_{\epsilon_k R_k}(0)$ then $|\nabla v_k(s,t)| \leq 2$ for all $(s,t) \in D_k$. Finally, since $u_k \in \mathcal{T}(a,b)$,

$$\partial_s v_k + J(v_k)\left(\partial_t v_k + \frac{1}{N_k} X\left(t_k + \frac{1}{N_k}, v_k\right)\right) = 0.$$

Now let $U_k$ denote the disk $B_{\epsilon_k}(0)$. Then we compute

$$\begin{aligned}
\int_{D_k} |\nabla v_k|^2 &= \int_{U_k} |\nabla u_k|^2 \\
&\leq \int_{U_k} \left(|\partial_s u|^2 + |\partial_t u_k - X(t,u_k) + X(t,u_k)|^2\right) dsdt \\
&\leq \int_{U_k} \left(\left(|\partial_s u_k|^2 + |\partial_t u_k - X(t,u_k)|^2\right) + |X(t,u_k)|^2 + 2\,|\partial_t u_k - X(t,u_k)| \cdot |X(t,u_k)|\right) dsdt \\
&\leq \int_{U_k} \left(\left(|\partial_s u_k|^2 + 2\,|\partial_t u_k - X(t,u_k)|^2\right) + 2\,|(t,u_k)|^2\right) dsdt \\
&\leq 2\int_{\mathbb{R}\times S^1} \left(|\partial_s u_k|^2 + |\partial_t u_k - X(t,u_k)|^2\right) dsdt + 2\int_{U_k} |X(t,u_k)|^2 \, dsdt \\
&= 2\int_{\mathbb{R}} |\partial_s (u_k)_s| + |\partial_t (u_k)_s - X(t,(u_k)_s)|^2 \, ds + 2\int_{U_k} |X(t,u_k)|^2 \, dsdt \\
&= 4\int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{A}((u_k)_s) \, ds + 2\int_{U_k} |X(t,u_k)|^2 \, dsdt \\
&= 4\,(\mathcal{A}(a) - \mathcal{A}(b)) + 2r_k,
\end{aligned}$$

where $r_k \to 0$ as $k \to \infty$, and as before $(u_k)_s(t) = u_k(s,t)$. Thus there exists a constant $C \geq 0$ such that

$$\int_{D_k} |\nabla v_k|^2 \leq C \text{ for all } k \in \mathbb{N}.$$

Thus by Theorem 4.16 we deduce the existence of a $J$-holomorphic curve $v \in C^\infty(\mathbb{R}^2, T^*M)$ and a subsequence $(v_{k_j})$ such that $v_{k_j} \to v \in C^\infty_{\mathrm{loc}}(\mathbb{R}^2, T^*M)$. Note also that $v$ satisfies $|\nabla v(0,0)| = 1$ (so $v$ is not constant) and

$$\int_{\mathbb{R}^2} |\nabla v|^2 \leq C.$$

Hence $v$ has finite energy, and so by the Removable Singularity Theorem 4.15 we conclude that $v$ extends to a non-constant $J$-holomorphic sphere $\tilde{v} : S^2 \to T^*M$, contradicting Lemma 4.14. ■

The proof of Theorem 4.12 is now easy.

*Proof. (of the Precompactness Theorem* 4.12*)*

Theorem 4.11 and Proposition 4.15 allow us to conclude by the Arzelà-Ascoli Theorem that $\mathcal{T}(a,b)$ has compact closure in $C^0_{\mathrm{loc}}(\mathbb{R} \times S^1, T^*M)$. Then we apply elliptic regularity to deduce that all solutions of

$\bar{\partial}_{H,J}(u) = 0$ are smooth, any $C^0$-limit of solutions is a solution and both the $C^0$ and $C^\infty$ topologies coincide on $\mathcal{T}(a,b)$. Hence $\mathcal{T}(a,b)$ is $C^0_{\text{loc}}$-precompact and hence $C^\infty_{\text{loc}}$-precompact. ∎

**Constructing the Floer complex $CF_*(T^*M, \omega, H, J)$**

For this subsection we let $H$ be a non-degenerate Hamiltonian of the form (3.10), and let $J \in \mathcal{J}_{j_0,\text{reg}}(H)$. Thus the trajectory spaces $\mathcal{T}(a,b,H,J)$ are all smooth manifolds of dimension $\mu(a) - \mu(b)$ that are precompact in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1, T^*M)$ for all $a, b \in \mathcal{P}(H)$. They are never compact though; in fact they admit a natural smooth free $\mathbb{R}$-action.

**Definition 4.18.** Let $a, b \in \mathcal{P}(H)$. We can define a natural smooth free $\mathbb{R}$-action on $\mathcal{T}(a,b)$ by

$$\alpha \cdot u(s,t) := u(s + \alpha, t).$$

We denote by $\mathcal{M}(a,b)$ the **moduli space** $\mathcal{T}(a,b)/\mathbb{R}$. $\mathcal{M}(a,b)$ is a smooth manifold of dimension $\mu(a) - \mu(b) - 1$. Note that this shows that $\mathcal{T}(a,b) = \emptyset$ if $\mu(a) = \mu(b)$.

We now define convergence in the moduli spaces.

**Definition 4.19.** A sequence $(U_n) \in \mathcal{M}(a,b)$ converges to $U \in \mathcal{M}(a,b)$ if for any lifts $(u_n)$ of the $(U_n)$ and any lift $u$ of $U$ there exists $(s_n) \in \mathbb{R}$ such that

$$u_n(s + s_n, t) \to u(s,t)$$

in the topology of $\mathcal{T}(a,b)$ (given as a submanifold of the Banach manifold $\mathcal{B}(a,b)$).

The proof of the following lemma is omitted; a proof can be found in [Ke], Lemma 1.5.6.

**Lemma 4.20.** *Suppose $u_n, u \in \mathcal{T}(a,b)$ such that $u_n \to u$ in the $C^\infty_{loc}$-topology. Then $u_n \to u$ with respect to the usual topology of $\mathcal{T}(a,b)$.*

The next result is the analogous version of Theorem 2.10; as we have already proved one 'broken orbit' style theorem, we omit the proof - essentially it follows from the Precompactness Theorem 4.12 in the same way that the Broken Orbit Lemma Theorem 2.10 followed from Corollary 2.9.

**Theorem 4.21. (The Broken Orbit Lemma for Floer homology)**
*Let $a, b \in \mathcal{P}(H)$ with $\mu(a) \geq \mu(b) + 1$. Given any sequence $(u_n) \in \mathcal{T}(a,b)$, there exists a subsequence $(u_{n(k)})$, critical points $a_0 = a, a_1, a_2, \dots, a_m = b \in \mathcal{P}(H)$ $(m \geq 1)$ with $\mu(a_i) > \mu(a_{i+1})$ and subsequences $(s_{(k,i)})_{k=1}^\infty$ such that*

$$v_{k,i}(s,t) := u_{n(k)}(s + s_{(k,i)}, t)$$

*converges in $C^\infty_{loc}(\mathbb{R} \times S^1, T^*M)$ to some $u_i \in \mathcal{T}(a_i, a_{i+1})$ (for $i = 0, \dots, m-1$) . In this case we say $u_{n(k)}$ converges **weakly** to each of the $u_i$.*

*We call the $u_i$ **broken trajectories** from $a$ to $b$. Letting $U_i$ denote the equivalence class of $u_i$ in $\mathcal{M}(a,b)$, we write*

$$U_{n(k)} \rightsquigarrow (U_0, \dots, U_{m-1})$$

*and say that $U_{n(k)}$ converges **weakly** to the m-tuple $(U_0, \dots, U_{m-1})$.*

This allows us to abstractly compactify the moduli space.

**Definition 4.22.** Given $a, b \in \mathcal{P}(H)$ we define the **compactified moduli space**

$$\widehat{\mathcal{M}(a,b)} = \mathcal{M}(a,b) \coprod_{k \geq 1} \mathcal{M}(a,b)_k,$$

where

$$\mathcal{M}(a,b)_k := \bigcup_{a,b \neq a_1 \neq \cdots \neq a_k \in \mathcal{P}(H)} \mathcal{M}(a,a_1) \times \cdots \times \mathcal{M}(a_k, b),$$

and we *define* convergence in $\widehat{\mathcal{M}(a,b)}$ as the weak $\rightsquigarrow$ convergence from Theorem 4.21. It is immediate from Theorem 4.21 that with this definition of convergence $\widehat{\mathcal{M}(a,b)}$ is indeed compact, and Lemma 4.20 shows this new type of convergence extends the original notion of convergence in $\mathcal{M}(a,b)$.

Next we move on to Floer's Gluing Theorem. This is essentially a converse result to the precompactness result. We have shown that we can compactify $\mathcal{M}(a,b)$ by adding in the broken trajectories. The Gluing Theorem will prove that every possible broken trajectory must appear in the compactification, and that near to each broken trajectory, the compactification has the structure of a smooth manifold with corners. In fact, we only need a special case of the Gluing Theorem (that is, for broken trajectories that break in one place only) to construct the Floer complex, and we shall content with stating only this special case.

**Theorem 4.23. (Floer's Gluing Theorem)**
*Let $a,b,c \in \mathcal{P}(H)$ be critical points with $\mu(a) > \mu(b) > \mu(c)$. Then given a compact subset $K \subseteq \mathcal{T}(a,b) \times \mathcal{T}(b,c)$ there exists a constant $C = C(K) > 0$ and a smooth* ***gluing map***

$$\mathcal{G} : K \times [C,\infty) \to \mathcal{T}(a,c), \qquad (u,v,R) \mapsto \mathcal{G}_R(u,v)$$

*such that $\mathcal{G}_R : K \hookrightarrow \mathcal{T}(a,c)$ is an embedding for all $R > C$. Moreover, given a compact set $L \subseteq \mathcal{M}(a,b) \times \mathcal{M}(b,c)$, $\mathcal{G}$ induces a smooth embedding*

$$\tilde{\mathcal{G}} : L \times [C,\infty) \hookrightarrow \mathcal{M}(a,c)$$

*such that $\tilde{\mathcal{G}}_R(U,V) \rightsquigarrow (U,V)$ as $R \to \infty$. Finally if $(u_n) \in \mathcal{T}(a,c)$ are such that there exist $U \in \mathcal{M}(a,b)$ and $V \in \mathcal{M}(b,c)$ such that $U_n \rightsquigarrow (U,V)$ (where $U_n$ is the image of $u_n$ in $\mathcal{M}(a,c)$) then for $n$ large enough the $U_n$ are in the image of $\tilde{\mathcal{G}}$.*

*Proof.* We shall only really outline the proof; full details can be found in [Sc1], §2.5 or [Sa], §3.3. The essential idea is to use the broken trajectories to construct an approximate solution of Floer's equation (4.2) running from $a$ to $c$, and then use an infinite dimensional implicit function theorem to show that there must be an actual solution as well.

Let us begin with the construction of the approximate solution - this is called **pregluing**. Recall by definition of $\mathcal{B}(a,b)$ given $u \in \mathcal{T}(a,b)$ we have some $s_0 \geq 0$ such that

$$u(s,t) = \begin{cases} \exp_{a(t)}(\alpha(s,t)) & s \leq -s_0 \\ \exp_{b(t)}(\beta(s,t)) & s \geq s_0. \end{cases}$$

Similarly if $v \in \mathcal{T}(b,c)$ there exists some $s_1 \geq 0$ such that

$$v(s,t) = \begin{cases} \exp_{b(t)}(\gamma(s,t)) & s \leq -s_1 \\ \exp_{c(t)}(\delta(s,t)) & s \geq s_1. \end{cases}$$

Select a cutoff function $\phi : \mathbb{R} \to [0,1]$ such that $\phi(s) \equiv 1$ for $s \geq 1$ and $\phi(s) \equiv 0$ for $s \leq 0$. Now for $R > \max\{s_0, s_1\}$, we define the pregluing map $\mathcal{G}^0$ by:

$$\mathcal{G}^0_R(u,v)(s,t) := \begin{cases} u(s+R,t) & s \leq -R/2-1 \\ \exp_{b(t)}(\phi(-s-R/2)\beta(s,t)) & -R/2-1 \leq s \leq -R/2 \\ b(t) & -R/2 \leq s/ \leq R/2 \\ \exp_{b(t)}(\phi(s-R/2)\gamma(s,t)) & R/2 \leq s \leq R/2+1 \\ v(s-R,t) & s \geq R/2+1. \end{cases}$$

Next one checks that $\mathcal{G}^0_R(u,v)$ lies in $\mathcal{B}(a,c)$ and converges to the broken trajectory $(u,v)$ as $R \to \infty$. Then the much harder step is to show that for $R$ sufficiently large $\mathcal{G}^0_R(u,v)$ corresponds to an actual solution $w := \mathcal{G}_R(u,v) \in \mathcal{T}(a,c)$. Let $D_R := D_{\mathcal{G}^0_R(u,v)}$ denote the linearised operator, as in the proof of Theorem 4.8. One shows that for large enough $R$, $D_R$ is surjective and admits a right inverse which is uniformly bounded independently of $R$.

The following **zero point lemma** is taken from [Sc1], Lemma 2.52:
*Let $F : V \to W$ be a smooth map between Banach spaces of the form*

$$F(x) = F(0) + DF_0(x) + F_1(x)$$

*for $x$ close to $0 \in V$. Suppose that $DF_0$ has a finite dimensional kernel, and a right inverse $G : W \to V$ such that there exists $C \geq 0$ such that*

$$\|GF_1(x) - GF_1(y)\| \leq C(\|x\| + \|y\|)\|x-y\| \text{ for all } x,y \in B_{\frac{C}{5}}(0), \text{ and } \|GF(0)\| \leq \frac{C}{10}.$$

*Then there exists a unique $x_0 \in B_{C/5}(0) \cap G(W)$ such that $F(x_0) = 0$. Moreover, $\|x_0\| \leq 2\|GF(0)\|$.*

With a lot of effort we then establish estimates that allow us to apply this result to the linearised operator $D_R$, and this thus yields a unique element $\mathcal{G}_R(u,v) \in \mathcal{T}(a,c)$ close to $\mathcal{G}^0_R(u,v)$. Then one must still prove the embedding property, and then investigate how the gluing map factors to the moduli spaces. The reader is referred to [Sc1], §2.5 for this tricky topic. ■

This next result is an immediate corollary and is analogous to Proposition 2.12. We haven't seen a version of the second statement crop up in Morse theory; if however we had proved the Morse Boundary Homomorphism Theorem 2.19 then we would have needed a result similar to this for the spaces $\mathcal{W}(x, y)$ when $m(x) - m(y) = 2$.

**Corollary 4.24.** *If $\mu(a) - \mu(b) = 1$ then $\mathcal{M}(a, b)$ consists of finitely many points. If $\mu(a) - \mu(b) = 2$ then $\widehat{\mathcal{M}(a, b)}$ consists of finitely many closed intervals and circles.*

*Proof.* If $\mu(a) - \mu(b) = 1$ then $\mathcal{M}(a, b)$ is a zero dimensional manifold, and thus there are no broken connecting orbits from $a$ to $b$. Hence $\mathcal{M}(a, b) = \widehat{\mathcal{M}(a, b)}$. Thus $\mathcal{M}(a, b)$ is compact, and hence a finite set. If $\mu(a) - \mu(b) = 2$ then given a broken trajectory connecting $a$ to $b$, via a critical point $c \in \mathcal{P}(H)$, the moduli spaces $\mathcal{M}(a, c)$ and $\mathcal{M}(c, b)$ are zero dimensional, hence discrete by the above, whence the last statement of the Gluing Theorem implies that the images of the gluing maps are precisely the ends of the one dimensional moduli space $\mathcal{M}(a, b)$. It follows that

$$\partial \widehat{\mathcal{M}(a, b)} = \bigcup_{c \in \mathcal{P}_k(H)} \mathcal{M}(a, c) \times \mathcal{M}(c, b)$$

and $\widehat{\mathcal{M}(a, b)}$ is a compact one dimensional manifold with (possibly empty) boundary. Finally, by Lemma 3.15 there are only finitely many critical points $c$ such that there are broken trajectories connecting $a$ to $c$ and $c$ to $b$, and thus $\widehat{\mathcal{M}(a, b)}$ consists of finitely many closed intervals and circles. ■

Now we can finally give the definition of the Floer complex: note the close similarity to the definition of the Morse complex. Let $\mathcal{P}_k(H)$ denote the set of critical points of $\mathcal{A}$ of index $\mu(a) = k$.

**Definition 4.25.** Let $a, b \in \mathcal{P}(H)$ satisfy $\mu(a) - \mu(b) = 1$. Define $n(a, b)$ to be the number of connected components of $\mathcal{M}(a, b)$, taken mod 2.

Let $CF_k = CF_k(T^*M, H, J)$ denote the free abelian group of critical points $a \in \mathcal{P}_k(H)$, and define $\partial^F : CF_k \to CF_{k-1}$ by

$$\partial^F(a) = \sum_{b \in \mathcal{P}_{k-1}(H)} n(a, b)b, \qquad a \in CF_k.$$

We call $CF_* = CF_*(T^*M, \omega, H, J)$ the **Floer complex** of $T^*M$ with respect to $\omega, H$ and $J$.

Of course, having made this definition we need to prove:

**Theorem 4.26.** $\partial^F \circ \partial^F = 0$

*Proof.* This is immediate from Corollary 4.24. Indeed, a compact one dimensional manifold with boundary has an even number of boundary points. That is, if $b \in \mathcal{P}_{k-1}(H)$ contributes to $\partial^F_k \circ \partial^F_{k+1}(a)$ then there exists a broken orbit $(a, c, b)$ for some $c \in \mathcal{P}_k(H)$. It corresponds to an endpoint of some interval in $\widehat{\mathcal{M}(a, b)}$, and the other endpoint of this interval cancels the contribution of $b$ to $\partial^F_k \circ \partial^F_{k+1}(a)$ mod 2. ■

Thus $CF_*$ defines a chain complex, and the associated homology $HF_*(T^*M, \omega, H, J)$ is the **Floer homology** of $T^*M$ with respect to $\omega, H$ and $J$.

*Remark.* In fact, the Floer homology of $T^*M$ is independent of the choice of $H$ and $J$ (provided they satisfy the required assumptions); this is **Floer's Continuation Priniciple** and is one of the great strengths of Floer homology. This can of course be shown directly (we refer the reader to [AS], Theorem 1.12 and Theorem 1.13); we however will prove this indirectly in the next section. We remark however that the chain complex $CF_*(T^*M, \omega, H, J)$ is **not** independent of the choice of $H$ and $J$ (in fact, changing $J$ yields isomorphic chain complexes, but changing $H$ produces chain homotopic complexes; see [AS], Theorem 1.12 and Theorem 1.13), and the isomorphism we construct in the next section will actually be on the chain complex level, and thus will itself depend on $H$ and $J$.

We conclude this section by constructing the Floer complex $HF_*(T^*M, \Omega, H, J)$. Given our work in §3 and the above, this is simple. The remark after Proposition 3.10 shows that if we could repeat the entire argument above with $H^{\mathrm{tw}}$ (so that $HF_*(T^*M, \omega, H^{\mathrm{tw}}, J)$ is well defined) instead of $H$ then we would have

$$HF_*(T^*M, \Omega, H, J) \cong HF_*\left(T^*M, \omega, H^{\mathrm{tw}}, J\right).$$

There are three points in the above argument where it is not obvious the results will continue to hold for $H^{\text{tw}}$ instead of $H$; namely the Index Theorem 4.2, the Trajectory Space Theorem 4.10 and the $L^\infty$-gradient estimates of Theorem 4.12. Luckily however in all three cases the original sources ([W2], Theorem 1.2 for Theorem 4.2, [FHS], Theorem 5.1 for Theorem 4.10 and [AS], Theorem 1.9 for Theorem 4.12) establish their results for a more general class of Hamiltonians then the ones we are considering, and in all three cases Hamiltonians of the form $H^{\text{tw}}$ are easily seen to be covered.

## 5. The isomorphism between the Morse and Floer complexes

### The statement of the theorem

In this final section we state and outline the proof of the main result of this paper, due to Abbandandolo and Schwarz; namely the isomorphism between $CF_*(T^*M, \omega, H, J)$ and $CM_*(\mathcal{L}M, L, \mathcal{G})$. We will closely follow the argument in [AS], §3, but will omit some of the technical difficulties.

The precise statement is the following, where we will define $\mathcal{J}^+_{j_1,\text{reg}}(H)$ later in this section.

**Theorem 5.1. (The Isomorphism between the Morse and Floer complexes)**
*Let H be a non-degenerate Hamiltonian of the form* (3.10)*, and L the (inverse) Legendre transformation of H. Let $J \in \mathcal{J}^+_{j_1,reg}(H)$, and let $\mathcal{G}$ be a Morse-Smale metric for L on the loop space $\mathcal{L}M$. Then there exists a chain complex isomorphism*

$$\Theta : CM_*(\mathcal{L}M, L, \mathcal{G}) \to CF_*(T^*M, \omega, H, J)$$

*of the form*

$$\Theta(x) = \sum_{\mu(a)=m(x)} N(x,a)a, \qquad x \in \mathcal{P}(L)$$

*where $N(x,a) = 0$ if $\mathcal{S}(x) \le \mathcal{A}(a)$, unless x and a correspond to the same solution (that is, $\dot{a} \doteqdot \left(\dot{x}, \dot{x}^\flat\right)$), in which case $N(x,a) = \pm 1$.*

The Index Theorem 4.2 shows us that the chain groups $CM_k$ and $CF_k$ coincide. This of course is not enough however. To define a chain homomorphism fix some $x \in \mathcal{P}(L)$ and $a \in \mathcal{P}(H)$ and fix some $r \in (2,4]$. Now consider the **half trajectory space**[12]

$$\mathcal{H}(x,a) \subseteq C^\infty(\mathbb{R}_{>0} \times S^1, T^*M) \cap W^{1,r}\left((0,1) \times S^1, T^*M\right)$$

given by those maps $u \in C^\infty(\mathbb{R}_{>0} \times S^1, T^*M) \cap W^{1,r}\left((0,1) \times S^1, T^*M\right)$ satisfying[13] the **modifed Floer equations**:

$$\bar{\partial}^+_{H,J}(u) := \partial_s u - J_t(u)\left(\partial_t u - X_H(t,u)\right) = 0 \text{ on } \mathbb{R}_{>0} \times S^1, \tag{5.1}$$

$$\tau \circ u_0(t) \in W^u(x), \qquad \lim_{s\to\infty} u(s,t) = a(t) \text{ uniformly in } t, \tag{5.2}$$

where $\tau : T^*M \to M$ is the footpoint map and $W^u(x)$ denotes the unstable manifold of $x$, as in §2.

We now proceed as in §4. Under an appropriate further restriction on $J$, the spaces $\mathcal{H}(x,a)$ are finite dimensional smooth manifolds of dimension $m(x)-\mu(a)$. An analogue of the Precompactness Theorem will show $\mathcal{H}(x,a)$ is precompact in $C^\infty_{\text{loc}}\left(\mathbb{R}_{\ge 0} \times S^1, T^*M\right)$. Similarly there is a version of Floer's Gluing Theorem for the spaces $\mathcal{H}(x,a)$. Given this machinery, much like the actual construction of the Floer complex, the construction of the homomorphism $\Theta$ and the proof that $\Theta$ is an isomorphism is relatively simple.

Before we get started on the details of the proof, we present a lemma that we will need later.

**Lemma 5.2.** *The following two statements hold:*
(i) *If $a : S^1 \to T^*M$ is a curve $a(t) = (x(t), p(t))$ of class $W^{1,2}$ then*

$$\mathcal{A}(a) \le \mathcal{S}(x), \tag{5.3}$$

*with equality if and only if $a \in \mathcal{P}(H)$ (and so $x \in \mathcal{P}(L)$).*

[12]This is somehat of a misnomer.

[13]We need to impose the condition that $u \in W^{1,r}\left((0,1) \times S^1, T^*M\right)$ with $r \in (2,4]$ in order that the first boundary condition is well defined. Indeed, if $u \in W^{1,r}$ then $u_0 \in W^{1-1/r,r}$, and since $W^u(x)$ consists of curves of class $W^{1,2}$, we thus require $W^{1,2} \hookrightarrow W^{1-1/r,r}$. Now recall that the (fractional) Sobolev Embedding Theorem gives an embedding of $W^{k,s} \hookrightarrow W^{\ell,r}$ if $k \ge \ell$ and $k - n/r \ge \ell - n/s$. In this case $k = 1$, $\ell = 1 - 1/r$, $n = 1$ and $s = 2$; we are thus required to have $1 - 1/2 \ge 1 - 2/r$, and hence $r \le 4$. The condition $r > 2$ is required for the same reason as it was in §4.

(ii*) Given $x \in \mathcal{P}(L)$ and $a \in \mathcal{P}(H)$, if $\mathcal{S}(x) \leq \mathcal{A}(a)$ then $\mathcal{H}(x,a) = \emptyset$ unless $x$ and $a$ correspond to the same solution (that is, $\dot{a} \doteqdot (\dot{x}, \dot{x}^\flat)$), in which case $\mathcal{H}(x,a) = \{a\}$.*

*Proof.* (i) This follows directly from the definition of $\mathcal{A}$, $\mathcal{S}$ and the Legendre transformation. Indeed,

$$\mathcal{S}(x) = \int_0^1 \left(\frac{1}{2}|\dot{x}(t)|^2 - V(t,x(t))\right) dt,$$

$$\begin{aligned}\mathcal{A}(a) &= \int_0^1 p(t)(\dot{x}(t)) - H(t,x(t),p(t))dt \\ &= \int_0^1 p(t)(\dot{x}(t)) - \frac{1}{2}|p(t)|^2 - V(t,x(t))dt, \\ &\leq \int_0^1 |p(t)|\,|\dot{x}(t)| - \frac{1}{2}\,|p(t)|^2 - V(t,x(t)),\end{aligned}$$

and our assertion comes down to $2cd \leq c^2 + d^2$. Equality is achieved if and only if $p(t) = \dot{x}(t)^\flat$, that is, if and only if $a$ is a critical point, and thus if and only if $x$ is a critical point.

(ii) Suppose now $\mathcal{H}(x,a) \neq \emptyset$, and let $u \in \mathcal{H}(x,a)$. Then since

$$\mathcal{A}(u_s) = \mathcal{A}(a) + \int_s^\infty \int_0^1 |-J_t(u_s)\,(\dot{u}_s - X_H(t,u_s))|^2_{J_t}\, dtds \geq \mathcal{A}(a),$$

we conclude that $\mathcal{A}(a) \leq \mathcal{A}(u_s) \leq \mathcal{A}(u_0) \leq \mathcal{S}(\tau \circ u_0)$, and since the latter lies in the unstable manifold $W^u(x)$ and $\mathcal{S}$ is a Lyapunov function, we have $\mathcal{S}(\tau \circ u_0) \leq \mathcal{S}(x)$. Tracing through the case of equality through we conclude $\mathcal{A}(a) = \mathcal{S}(x)$ if and only if $\dot{a} \doteqdot \left(\dot{x}, \dot{x}^\flat\right)$. It is clear that in this case $\mathcal{H}(x,a)$ then consists of a single element, namely the stationary solution $u(s,t) = a(t)$. ■

**The Fredholm property, precompactness and gluing**

We first need to define a suitable modification $\mathcal{B}^+(x,a)$ of the previous Banach manifold $\mathcal{B}(a,b)$ from §4. Fix some $r \in (2,4]$ and let $\mathcal{B}^+(x,a)$ denote that set of maps $u : \mathbb{R}_{\geq 0} \times S^1 \to T^*M$ that are of class $W^{1,r}_{\text{loc}}(\mathbb{R}_{\geq 0} \times S^1, T^*M)$ that satisfy $\tau \circ u_0 \in W^u(x)$ and for which there exists some $s_0 \geq 0$ and a $W^{1,r}$ section $\alpha$ of the the bundle $a^*(TT^*M)$ (regarded as a bundle over $(s_0,\infty) \times S^1$) such that

$$u(s,t) = \exp_{a(t)}(\alpha(s,t)), \qquad (s,t) \in (s_0,\infty) \times S^1.$$

Similarly to $\mathcal{B}(a,b)$, $\mathcal{B}^+(x,a)$ can be given the structure of a Banach manifold with $T_u\mathcal{B}^+(x,a)$ identified with the space of $W^{1,r}$ sections $\sigma$ of $u^*(TT^*M)$ such that

$$D(\tau \circ u_0)(\sigma_0(t)) \in T_{\tau\circ u_0(t)} W^u(x), \qquad \text{for all } t \in S^1.$$

Next, we define $\hat{\mathcal{B}}^+(x,a) \to \mathcal{B}^+(x,a)$ to be the Banach bundle over $\mathcal{B}^+(x,a)$ whose fibre over $u \in \mathcal{B}^+(x,a)$ consists of the $L^r$ sections of $u^*(TT^*M)$. Then similarly to Propositon 4.7, we may extend $\bar{\partial}^+_{H,J}$ to a smooth section

$$\bar{\partial}^+_{H,J} : \mathcal{B}^+(x,a) \to \hat{\mathcal{B}}^+(x,a)$$

such that $\mathcal{H}(x,a) = \left(\bar{\partial}^+_{H,J}\right)^{-1}(0)$. See [AS], §3.1, p36 for more details on this construction.

The result we have in mind (compare to Theorem 4.7) is the following.

**Theorem 5.3. (The Fredholm property of $\bar{\partial}^+_{H,J}$)**

*Suppose $\mathcal{H}(x,a) \neq \emptyset$, and fix some $u \in \mathcal{H}(x,a)$. The linearisation $D^+_u = \left(D\bar{\partial}^+_{H,J}\right)_u : T_u\mathcal{B}^+(x,a) \to L^r(u^*(TT^*M))$ is a Fredholm operator of index $m(x) - \mu(a)$.*

The proof of the Fredholm property is similar to that of Proposition 4.7; the proof of the Fredholm index is somewhat harder - full details can be found in [AS], Theorem 3.4.

As before in order to conclude that $\mathcal{H}(x,a)$ is a submanifold of $\mathcal{B}^+(x,a)$ we need to know that 0 is a regular value of $\bar{\partial}^+_{H,J}$, or equivalently that the linearisation $D^+_u$ is surjective. Again sadly this is not always the case, but denoting by $\mathcal{J}^+_{\text{reg}}(H)$ the subset of $\mathcal{J}_{\text{reg}}(H)$ in which 0 is a regular value of $\bar{\partial}^+_{H,J}$ we have the following theorem:

**Theorem 5.4.** *$\mathcal{J}^+_{reg}(H)$ is residual in $\mathcal{J}_{reg}(H)$.*

In order to prove a Precompactness Theorem for the half trajectory spaces $\mathcal{H}(x,a)$ we need an extension of the $L^\infty$ gradient estimates given in Theorem 4.11. The following result is from [AS], Theorem 1.9.3. For this result we will assume that $M$ is a submanifold of $\mathbb{R}^N$, via an isometric embedding $M \hookrightarrow \mathbb{R}^N$ (use Nash's Theorem).

**Theorem 5.5.** **($L^\infty$ gradient estimates for $\mathcal{H}(x,a)$)**
*Given $r > 2$, there exists $j_1 = j_1(r) \le j_0$ such that if $J$ is a time dependent almost complex structure such that $\left\|J_t - J_g\right\|_\infty < j_1$ for all $t$ then given $c, d, e \in \mathbb{R}$, the set of solutions*

$$u(s,t) = (x(s,t), p(s,t)) \in C^\infty\left(\mathbb{R}_{>0} \times S^1, T^*M\right) \cap W^{1,r}\left((0,1) \times S^1, T^*M\right)$$

*of $\bar{\partial}^+_{H,J}(u) = 0$ such that*

$$c \le \mathcal{A}(u_s) \le d \text{ for all } s \in \mathbb{R}_{\ge 0},$$

*and*

$$\|x(0,\cdot)\|_{W^{1-1/r,r}(S^1,\mathbb{R}^N)} \le e$$

*is bounded in $L^\infty\left(\mathbb{R}_{\ge 0} \times S^1, T^*M\right)$.*

As with Theorem 4.12, we shall not try to prove this. Instead, we deduce the desired Precompactness Theorem:

**Theorem 5.6.** **(The Precompactness Theorem for $\mathcal{H}(x,a)$)**
*Suppose $J$ is a time dependent almost complex structure on $T^*M$ such that $\left\|J_t - J_g\right\| < j_1$ for all $t$. Then for every $x \in \mathcal{P}(L)$ and $a \in \mathcal{P}(H)$, the half trajectory space $\mathcal{H}(x,a)$ is precompact in $C^\infty_{loc}\left(\mathbb{R}_{\ge 0} \times S^1, T^*M\right)$.*

*Proof.* *(sketch)*
We use the fact that the unstable manifold $W^u(x)$ is precompact in the $W^{1,2}$ topology of $\mathcal{L}M$ by Corollary 2.11 to conclude that the $W^{1,2}$ norm of $\tau \circ u_0$ is uniformly bounded (since $\tau \circ u_0 \in W^u(x)$. Since $W^{1,2}((0,1))$ continuously embeds in $W^{1-1/r,r}((0,1))$ as $r \le 4$ (see the footnote on the previous page) we conclude that the $W^{1-1/r,r}((0,1))$ norm of $\tau \circ u_0$ is uniformly bounded. Theorem 5.5 then implies that $\mathcal{H}(x,a)$ is bounded in $L^\infty$. The rest of the argument then proceeds similarly to the Precompactness Theorem 4.12. ■

The argument now proceeds similarly to that of §4; one proves analogues of the Broken Orbit Lemma and the Gluing Theorem (precise statements can be found in [AS], Proposition 3.6 and Proposition 3.9). Finally we deduce the following analogue of Corollary 4.24.

**Corollary 5.7.** *If $x \in \mathcal{P}_k(L)$ and $a \in \mathcal{P}_k(H)$ then $\mathcal{H}(x,a)$ is a finite set. If instead $a \in \mathcal{P}_{k-1}(H)$ then $\mathcal{H}(x,a)$ consists of finitely many closed intervals and circles; moreover we can identify the boundary with*

$$\partial\mathcal{H}(x,a) \cong \left(\bigcup_{y\in\mathcal{P}_{k-1}(L)} \mathcal{H}(y,a) \times \mathcal{W}(x,y)/\mathbb{R}\right) \cup \left(\bigcup_{b\in\mathcal{P}_k(H)} \mathcal{M}(b,a) \times \mathcal{H}(x,b)\right). \tag{5.4}$$

## The isomorphism

Select $J \in \mathcal{J}^+_{j_1,\mathrm{reg}}(H)$. We construct an isomorphism $\Theta : CM_*(\mathcal{L}M, L, \mathcal{G}) \to CF_*(T^*M, \omega, H, J)$ as follows: given $x \in \mathcal{P}(L)$ and $a \in \mathcal{P}(H)$ such that $m(x) = \mu(a)$, the half trajectory space $\mathcal{H}(x,a)$ is a finite set. Let $N(x,a)$ denote its cardinality mod 2, and define $\Theta : CM_k \to CF_k$ by

$$\Theta(x) = \sum_{a\in\mathcal{P}_k(H)} N(x,a), \qquad x \in \mathcal{P}_k(L).$$

**Theorem 5.8.** *$\Theta$ is a chain homomorphism, that is, $\Theta \circ \partial^M = \partial^F \circ \Theta$.*

*Proof.* We use Corollary 5.7. The left-hand side of (5.4) has cardinality 0 taken modulo 2, as a compact one dimensional manifold has an even number of points on the boundary. Hence

$$\sum_{y\in\mathcal{P}_{k-1}(L)} N(y,a)m(x,y) = \sum_{b\in\mathcal{P}_k(H)} n(a,b)N(x,b),$$

and this is precisely what we are required to prove. ■

**Theorem 5.9.** *$\Theta$ is an isomorphism*

*Proof.* This is beautifully simple. Order the elements of $\mathcal{P}_k(L)$ and $\mathcal{P}_k(H)$ by increasing action, assigning any order to the subsets of critical points with the same action, although ensuring that corresponding elements of $\mathcal{P}_k(L)$ and $\mathcal{P}_k(H)$ recieve the same order. Now consider the (possibly infinite) matrix representing $\Theta$ in this basis. We claim this matrix is upper triangular, with all entries on the main diagonal equal to $\pm 1$. This of course completes the proof, as such a matrix is necessarily invertible.

Indeed, the entries below the main diagonal correspond to pairs $(x, a)$ with $\mathcal{S}(x) \leq \mathcal{A}(a)$. By Lemma 5.2(ii) this shows that $\mathcal{H}(x, a) = \emptyset$ (and so $N(x, a) = 0$). Similarly entries on the main diagonal are solutions that correspond to each other, and Lemma 5.2(ii) states that $\mathcal{H}(x, a) = \{a\}$, whence $N(x, a) = \pm 1$.

This completes the proof of the Isomorphism Theorem 5.1. ■

Finally, as in §4 all of the arguments in this section work with the twisted Hamiltonian $H^{\mathrm{tw}}$ instead, and thus we also deduce the existence of an isomorphism between $CM_*(\mathcal{L}M, L^{\mathrm{tw}}, \mathcal{G})$ and $CF_*(T^*M, \omega, H^{\mathrm{tw}}, J)$, and thus also an isomorphism between the Floer homology of $T^*M$ under the twisted symplectic form $\Omega$ and the singular homology of $\mathcal{L}M$,

$$HF_*(T^*M, \Omega, H, J) \cong H_*^{\mathrm{sing}}(\mathcal{L}M).$$

## Appendix A. The proof of (3.17)

We want to show

$$\sum_{i,j,k} \left( \frac{1}{2} \partial_k g^{ij} + g^{i\ell} \Gamma^j_{\ell k} \right) = 0.$$

Starting from

$$g_{ij} g^{jm} = \delta^m_i,$$

we begin by differentiating both sides to obtain

$$\partial_k g_{ij} g^{jm} = -g_{ij} \partial_k g^{jm},$$

and then multiplying by $g^{i\ell}$ and summing over $\ell$,

$$\partial_k g_{ij} g^{jm} g^{i\ell} = -\partial_k g^{jm} g_{ij} g^{i\ell} = \partial_k g^{jm} \delta^\ell_j = \partial_k g^{\ell m},$$

and thus

$$\frac{1}{2} \partial_k g^{ij} = -\frac{1}{2} g^{i\ell} g^{jm} \partial_k g_{\ell m}. \tag{A.1}$$

Now recall by definition

$$\Gamma^j_{\ell k} = \frac{1}{2} g^{jm} \left( \partial_k g_{\ell m} + \partial_\ell g_{km} - \partial_m g_{\ell k} \right),$$

and so

$$g^{i\ell} \Gamma^j_{\ell k} = \frac{1}{2} g^{i\ell} g^{jm} \left( \partial_k g_{\ell m} + \partial_\ell g_{km} - \partial_m g_{\ell k} \right), \tag{A.2}$$

and thus combining (A.1) and (A.2) we obtain

$$\frac{1}{2} \partial_k g^{ij} + g^{i\ell} \Gamma^j_{\ell k} = \frac{1}{2} g^{i\ell} g^{jm} \left( \partial_\ell g_{km} - \partial_m g_{\ell k} \right).$$

Summing both sides over $i$, $j$ and $k$ then gives

$$\sum_{i,j,k} \left( \frac{1}{2} \partial_k g^{ij} + g^{i\ell} \Gamma^j_{\ell k} \right) = 0,$$

since all the terms on the right-hand side cancel.

## References


[AM] A. Abbondandolo and P. Majer, *Lectures on the Morse complex for infinite dimensional manifolds*, Morse Theoretic Methods in Nonlinear Analysis and Symplectic Topology, NATO Science Series II, **217**, Springer, New York-Heidelburg-Berlin, 2006.

[AS] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math., **59**, pp254-316, 2006.

[AL] M. Audin and J. Lafontaine (eds.), *Holomorphic curves in symplectic geometry*, Progress in Math. **117**, Birkhäuser, Basel-Boston-Berlin, 1994.

[Ben] V. Benci, *Periodic solutions of Lagrangian systems on a compact manifold*, J. Diff. Eq. **63**, pp135-161, 1986.

[Bes] A. Besse, *Manifolds all of whose Geodesics are Closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **93**, Springer, New York-Heidelburg-Berlin, 1978.

[BH] A. Banyaga and D. Hurtubise, *Lectures on Morse Homology*, Kluwer Texts Math. Sciences, **29**, Kluwer, 2004.

[E] L. Evans, *Partial Differential Equations*, Grad. Studies Math., **19**, AMS, Providence, Rhode Island, 1998.

[FHS] A. Floer, H. Hofer and D. Salamon, *Tranversality in elliptic Morse theory for the symplectic action*, Duke Math. J. **80**, pp252-292, 1996.

[G] V. Ginsburg, *On closed trajectories of a charge in a magnetic field. An application of symplectic geometry*, Contact and Symplectic Geometry (C. Thomas, eds.), Pub. of the Issac Newton Ins., Cambridge Uni., Press, 1996, pp131-148.

[HZ] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced Texts, Birkhäuser, Basel-Boston-Berlin, 1994.

[J] J. Jost, *Riemannian Geometry and Geometric Analysis 4th ed.*, Universitext, Springer, New York-Heidelburg-Berlin, 2005.

[Ke] E. Kerman, *From Morse homology to Floer homology*, lecture notes available from http://www.math.uiuc.edu/~ekerman/mf.html, 2005.

[K] W. Klingenberg, *Riemannian Geometry 2nd ed.*, Walter de Gruyter & Co., Berlin 1995.

[Lan] S. Lang, *Fundamentals of Differential Geometry,* Grad. Texts Math., **191**, Springer, New York-Heidelburg-Berlin, 1999.

[Lau] F. Laudenbach, *Symplectic geometry and Floer homology*, Symplectic geometry and Floer homology: a survey of Floer homology for manifolds with contact type boundary or symplectic homology, Ensaios Mat. **7**, pp1-50, Soc. Brasil. Mat., Rio de Janeiro, 2004.

[MS1] D. McDuff and D. Salamon, *J-Holomorphic Curves and Quantum Cohomology*, Uni. Lecture Series, **6** , AMS, Providence, Rhode Island, 1995.

[MS2] D. McDuff and D. Salamon, *Introduction to Symplectic Topology, 2nd ed.*, Oxford Math. Monographs, Oxford Uni. Press, Oxford, 1998.

[MS3] McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic topology*, Colloquium Publications, **52**, AMS, Providence, Rhode Island, 2004.

[RS] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27**, p1-33, 1995.

[R] J. Rotman, *An Introduction to Algebraic Topology*, Grad. Texts Math., **119**, Springer, New-York-Heidelburg-Berlin, 1988.

[Sa] D. Salamon, *Lectures on Floer Homology*, Symplectic geometry and topology (Y. Eliashberg and L. Trayner, eds.), IAS/Park City Mathematics Series, Amer. Math. Soc., pp143-225, 1999.

[SW] D. Salamon and J. Weber, *Floer homology and the heat flow*, GAFA **16**, pp1050-138, 2006.

[SZ] D. Salamon and E. Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math. **45**, pp1303-1360, 1992.

[Sc1] M. Schwarz, *Morse Homology*, Progress in Math. **111**, Birkhäuser, Basel-Boston-Berlin, 1993.

[Sc2] M. Schwarz, *Cohomology operations from $S^1$-cobordisms in Floer homology,* Ph.D. thesis, Swiss Federal Inst. of Techn. Zurich, Zurich, Diss. ETH No. 11182, 1995.

[Sc3] M. Schwarz, *Introduction to Symplectic Floer Homology*, Contact and Symplectic Geometry (C. Thomas, eds.), Pub. of the Issac Newton Ins., Cambridge Uni., Press, 1996, pp131-148.

[V] C. Viterbo, *Functors and computations in Floer homology with applications, Part II*, preprint, 1996.

[W1] J. Weber, *J-holomorphic curves in cotangent bundles and the heat flow*, Ph.D. thesis / Dissertation, TU Berlin, 1999

[W2] J. Weber, *Perturbed closed geodesics are periodic orbits: Index and Transverality*, Math. Z. **241** no.1, pp45-82, 2002.

[W3] J. Weber, *Three approaches towards Floer homology of cotangent bundles*, J. Symplectic Geom. **3** no.4, pp671-701, 2005.