Differentiation

1 Three Rules

There are three rules that we commonly use when differentiating polynomials. Firstly, the first derivitive of the nth power of x with respect to x is given by

$$\frac{d[x^n]}{dx} = nx^{n-1} \tag{1}$$

Secondly, we can deal with constants by simply moving them outside of the derivitive, i.e.

$$\frac{d[af(x)]}{dx} = a\frac{df(x)}{dx} \tag{2}$$

And thirdly, we can differentiate terms in a sum seperately

$$\frac{d[f(x)+g(x)]}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx} \tag{3}$$

2 The First Rule

The definition of the derivitive is

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \tag{4}$$

and thus the derivitive of the nth power of x is

$$\frac{d[x^n]}{dx} = \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right] \tag{5}$$

By the binomial theorum

$$(x+\Delta x)^n = x^n + \frac{n!}{(n-1)!1!}x^{n-1}\Delta x + ... + \Delta x^n \tag{6}$$

We can simplify the second term by noting

$$\frac{n!}{(n-1)!1!} = \frac{n \times (n-1) \times (n-2) \times ... \times 2}{(n-1) \times (n-2) \times ... \times 2} = n \tag{7}$$

Substituting this into equation 6, and equation 6 back into equation 5, we get

$$\begin{aligned}\frac{d[x^n]}{dx} &= \lim_{\Delta x \to 0}\left[\tfrac{x^n + nx^{n-1}\Delta x + ... + \Delta x^n - x^n}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0}\left[nx^{n-1} + ... + \Delta x^{n-1}\right] \\ &= nx^{n-1} \end{aligned} \tag{8}$$

3 The Second Rule

From the definition of differentiation

$$\begin{aligned}\frac{d[af(x)]}{dx} &= \lim_{\Delta x \to 0}\left[\tfrac{af(x+\Delta x) - af(x)}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0}\left[a\tfrac{f(x+\Delta x) - f(x)}{\Delta x}\right] \end{aligned} \tag{9}$$

Using the rule $\lim_{\Delta x \to 0}[a \times b] = \lim_{\Delta x \to 0}[a] \times \lim_{\Delta x \to 0}[b]$, we have

$$\frac{d[af(x)]}{dx} = \lim_{\Delta x \to 0}[a] \times \lim_{\Delta x \to 0}\left[\tfrac{f(x+\Delta x) - f(x)}{\Delta x}\right] \tag{10}$$

$$= a\tfrac{df(x)}{dx} \tag{11}$$

4 The Third Rule

From the definition of differentiation

$$\begin{aligned}\frac{d[f(x)+g(x)]}{dx} &= \lim_{\Delta x \to 0}\left[\tfrac{f(x+\Delta x) + (g(x) - (f(x) + g(x))}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0}\left[\tfrac{f(x+\Delta x) - f(x)}{\Delta x} + \tfrac{g(x+\Delta x) - g(x)}{\Delta x}\right] \end{aligned} \tag{12}$$

Using the rule $\lim_{\Delta x \to 0}[a + b] = \lim_{\Delta x \to 0}[a] + \lim_{\Delta x \to 0}[b]$, we have

$$\begin{aligned}\frac{d[f(x)+g(x)]}{dx} &= \lim_{\Delta x \to 0}\left[\tfrac{f(x+\Delta x) - f(x)}{\Delta x}\right] + \lim_{\Delta x \to 0}\left[\tfrac{g(x+\Delta x) - g(x)}{\Delta x}\right] \\ &= \tfrac{df(x)}{dx} + \tfrac{dg(x)}{dx} \end{aligned} \tag{13}$$