QB Christmas Exam Questions

Part A:
These should take around 15 minutes each

A1 t

(a) Sketch the following curves:

(i) $y(t) = y_0 e^{\beta t}$ for $t \geq 0$ (plot both curves on the same graph)

(1) if $\beta = 1$ and $y_0 = 1$
(2) if $\beta = 2$ and $y_0 = 1$

(ii) $y(t) = \kappa - (\kappa - y_0)e^{-\beta t}$ for $t \geq 0$ (plot both curves on the same graph)

(1) if $\kappa = 2$, $\beta = 1$ and $y_0 = 1$
(2) if $\kappa = 2$, $\beta = 0.5$ and $y_0 = 3$

(b) The growth of which populations can be described by (i) the exponential and (ii) monomolecular models? Give biological examples.

A2

OR

The concentration of molecules, $n(t)$, in a solution changes with time due to a chemical reaction according to the following rate equation: $dn/dt = \beta(\kappa - n)$.

(a) Find the solution of the above rate equation if the reaction starts at $t = 0$ when the concentration of molecules is $n(t = 0) = n_0 < \kappa$.

(b) What is the concentration, n_1, of molecules at time $t_1 = 1$ sec after the beginning of the reaction if $n_0 = 10^{20}$ cm^{-3}, $\kappa = 10^{21}$ cm^{-3} and $\beta = 0.5$ sec^{-1}?

(c) In a similar experiment, but with a different initial concentration of molecules, the concentration of molecules after $t_2 = 3$ sec has been measured and found to be $n_2 = 8 \times 10^{20}$ cm^{-3}. What is the initial concentration of molecules in this experiment if the values of κ and β are the same as in (b)?

Part B:
These should take around 30 minutes each

B1 Epidemics of some crop diseases can be described by a modified form of the logistic and monomolecular equations:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \left(\alpha P + \frac{\beta}{\kappa} I\right)(\kappa - I) \ ,$$

in which I are infected plants, P is inoculum that enters from outside the system, κ is the carrying capacity of the host population, and α and β are positive parameters for rates of disease transmission from external inoculum and infected plants, respectively.

(a) If t is measured in days and I in numbers of infected plants per unit area, what are the dimensions for α, β and κ?

(b) For the initial condition $I = 0$ at $t = 0$, and assuming that P and κ do not vary with time, show that the solution to the differential equation is

$$I = \frac{\kappa\left(1 - \mathrm{e}^{-mt}\right)}{1 + \gamma \mathrm{e}^{-mt}}, \qquad \text{where } m = \alpha P + \beta \text{ and } \gamma = \beta/\alpha P.$$

(c) Calculate the equilibrium points and establish whether or not any positive equilibrium is stable.

(d) Summarise the principal biological advantages and limitations of the logistic and monomolecular equations as models for biological growth.

B2 A substance dissolved in solution is pumped at volume flow rate Q down a semi-permeable tube of perimeter p. The substance passively diffuses across the wall of the tubing, and the rate of transfer per unit area of wall, j, is proportional to the concentration: $j = kC$. The concentration of the substance entering the tube is C_o.

(a) Use a local application of the equations for compartmental analysis to show that the steady-state concentration declines exponentially with distance x along the tube:

$$C(x) = C_o \, e^{\frac{-kp}{Q} x} \ .$$

(b) A very long tube will eventually absorb all of the substance. Show that the total rate of transfer over the length $x = 0$ to $x = L$ is given by

$$R = QC_o \left(1 - e^{\frac{-kp}{Q} L}\right).$$

(c) Sketch $R/(QC_o)$, which gives the fraction of the influx that has been absorbed by $x = L$.

(d) What drawbacks might there be for a very long tube? Briefly outline, in words, how an optimal length could be defined.

Merry Christmas!