Cochains

Contents

1 Introduction

This essay is a very short introduction to rational homotopy theory. Rational homotopy theory is an attempt to do homotopy theory “modulo torsion”. It turns out that it is possible to greatly simplify homotopy theory (which is notoriously hard) by abandoning all torsion information. Recall that the *torsion subgroup* of an abelian group is the subgroup of all elements of finite order. We can form the *rationalisation* of an abelian group by quotienting out the torsion subgroup. The first problem of rational homotopy theory is to find what the geometric analogue of this is.

The homotopy theory of rational spaces turns out to be very algebraic—we can actually capture the whole rational homotopy type of a (nice) space in one algebraic invariant, the so-called *minimal model*. The main theorem of this essay states a strong correspondence between connected nilpotent rational spaces of finite $\mathbb{Q}$-type up to homotopy and their minimal models. There is an equivalence of subcategories of the homotopy categories of topological spaces on the one hand and commutative differential graded algebras on the other hand.

The first characterisation of rational homotopy theory as algebraic was given by Quillen in [16], although he did not identify the convenient class of minimal model. The theory became clearer after Sullivan identified the polynomial de Rham complex of a space as the starting point for finding the algebraic analogue and introduced minimal models in [17]. It is these ideas that lie at the heart of the proof of our main result by Bousfield and Guggenheim [2]. While the method of proof is very clear by now there is a large number of technical details to be taken care of, especially if one is not restricting to the case of simply-connected spaces. The main goal of this essay is to provide as much of the proof for the main theorem as time and space permit. While all the essential ideas of the proof are presented, many details and some proofs of technical results were left out, the main reference for missing pieces is naturally [2]. Also, all necessary results about model categories, simplicial sets and spectral sequences are just quoted.

Let us describe the course of action: The second chapter will give the essential definitions of rational homotopy theory and one key example of a rational space. We will introduce minimal models with a number of examples in chapter 3. The main theorem relies on the notion of homotopy categories, which is developed in Quillen's theory of model categories. In chapter 4 we will establish this language and state some standard results which allow us to prove that minimal models are unique up to isomorphism. Chapter 5 starts with a very brief introduction to simplicial sets, after which we swap categories and talk about simplicial sets instead of topological spaces. This step is justified by the equivalence of the homotopy categories of simplicial sets and topological spaces. We then go on to construct the adjunction between simplicial sets and commutative differential graded algebras that will give rise to the equivalence of the main result. Finally, in chapter 6 we state and prove the main result (Theorem 6.1). The prove is by a form of induction and uses the fact that minimal models and nice simplicial sets can be obtained in step by step processes which are exactly dual. Chapter 7 is very much a dessert. With surprisingly little effort we prove that the minimal models (and thus rational homotopy types) of Kähler manifolds are formal consequences of their de Rham cohomology. The result is due to Deligne, Griffiths, Morgan and Sullivan [5].

We deviate from the Bousfield Guggenheim paper notably in two points. For one, we make more use of the homotopy theory of model categories, circumventing Proposition 5.7 of [2] whose proof does not seem to be correct, as Charley Crissman pointed out to me. The other point is that we have a new proof for the last step of the main proof. Again, we could not follow the original proof which seems to assume a stronger version of our Lemma 5.11 than is stated and proved in the paper.

I want to thank Ian Grojnowski for setting this essay and advising me. I am also grateful to Charley Crissman for some very fruitful discussions.

2 Rational Homotopy

A natural place to start looking for what “homotopy theory modulo torsion” might mean are the algebraic invariants of a space. Let’s recall that a map between connected CW-complexes is a homotopy equivalence if and only if it induces an isomorphism on all homotopy groups or (if the spaces are nilpotent) on all homology groups. The assumption of nilpotency will occur frequently in this essay. Nilpotent spaces are a generalisation of simply connected spaces; the definition will be postponed to chapter 6.

So let us consider spaces which are torsion-free on homology. Note that it is not enough to demand that there are no torsion elements in homology, since, the homotopy type is not determined by homology groups but by *maps on homology*. The integers for example have no elements of finite order, but there are maps $\mathbb{Z} \to \mathbb{Z}$ whose cokernel is torsion.

2.1 Rational Homotopy Type

A connected space X is *rational* if for $n \geq 1$, $\pi_n(X)$ is *uniquely divisible*, i. e. for every element $x \in \pi_n(X)$ and every $n \in \mathbb{Z}$ there is an element $y \in \pi_n(X)$ such that $x = y^n$. For a nilpotent space this is equivalent to defining X to be rational when for every $n \geq 1$, $H_n(X, \mathrm{pt}; \mathbb{Z})$ is a vector space over $\mathbb{Q}$, for a proof see [3]. Then a *rational homotopy equivalence* is a map $f : X \to Y$ such that $H_*(f; \mathbb{Q})$ is an isomorphism. A *rationalisation* of a nilpotent space X is a rational homotopy equivalence from X to a connected rational space $X_{\mathbb{Q}}$. We will later construct an explicit rationalisation functor for nilpotent spaces. In the non-nilpotent case we can still define a rationalisation in terms of a universal property: A *rationalisation* for a space X is a space $X_{\mathbb{Q}}$ and a map $X \to X_{\mathbb{Q}}$ such that every map $f : X \to Y$, where Y is rational, uniquely factors through $X_{\mathbb{Q}}$.

Two spaces whose rationalisation have the same weak homotopy type, i.e. can be connected by a chain of maps inducing isomorphisms on homotopy, have the same *rational homotopy type*.

2.2 Examples

Most spaces one finds in nature are far from being rational, so we should at least have a brief look at the simplest examples.

Example 2.1. The most elementary rational space is the *rational n-sphere* $S^n_{\mathbb{Q}}$. To construct it we start with a wedge $\bigvee_k S^n_k$ of n-spheres, one for each natural number k. Then attach $(n+1)$-discs D^{n+1}_k along the map $h : S^n \to S^n_k \vee S^n_{k+1}$ which is $[S^n_k] - (k+1)[S^n_{k+1}]$ on homology. That is, attach the disc to S^n_{k+1} and "wrap it up" $k+1$ times before attaching it to S^n_k. The homology of $S^n_{\mathbb{Q}}$ is $\mathbb{Z}$ in degree 0 and $\mathbb{Q}$ in degree n and 0 elsewhere. The inclusion of S^n in $S^n_{\mathbb{Q}}$ as S^n_1 is a rationalisation of the sphere.

Once we have constructed $S^n_{\mathbb{Q}}$, we can define the rational disc $D^n_{\mathbb{Q}} = S^n_{\mathbb{Q}} \times I / S^n_{\mathbb{Q}} \times \{0\}$. It is nontrivial, but not hard to believe, that with these building blocks we can build rational models for CW-complexes: We glue rational disks along rational spheres following similar instructions to those we had for building the original CW-complex. For a complete treatment of this approach see §9 of [7].

Example 2.2. Let us briefly demonstrate what kind of information we lose. Looking at the rational homology of the real projective space of dimension two it is immediate that $\mathbb{RP}^2 \to *$ is a rationalisation, i.e. the rational homotopy type of $\mathbb{RP}^2$ is trivial.

3 Differential Graded Algebras

As mentioned in the introduction, the main upshot of doing rational homotopy theory is that it is very algebraic, in a sense to be made precise now. We want to associate to a space an algebraic object that captures all the rational homotopy theory of the space. The homology groups or cohomology ring will not be enough, but we recall that they are themselves extracted from a larger object.

3.1 Basic definitions

Look at the singular cochain complex C^X or the de Rham complex Ω^*X of a space X; they both are *graded differential algebras* in the following sense.

Definition. Let k be a field. (We will mainly be interested in the case $k = \mathbb{Q}$.) A *graded algebra* is a graded k-module $A = \oplus_p A^p$ with a multiplication that is an associative linear map $\mu : A \otimes A \to A, x \otimes y \mapsto xy$ of degree zero, and a unit $\eta : k \to A$. Multiplication and unit must satisfy the appropriate identities $\mu(\mu \otimes \mathbf{1}) = \mu(\mathbf{1} \otimes \mu)$ and $\mu(\eta \otimes \mathbf{1}) = \mathbf{1}$.

The corresponding morphisms are of course morphisms $\phi : A \to B$ of graded modules that respect multiplication and unit: $\mu_B(\phi(x), \phi(y)) = \phi(\mu_A(x, y))$ and $\phi \circ \eta_A = \eta_b$.

A *differential graded algebra* is a graded algebra A with a *differential* that is a derivation. This means there is a linear map $d : A \to A$ of degree $+1$, satisfying $d^2 = 0$ and $d(fg) = (df)g - f(dg)$.

Example 3.1. For the de Rham complex the grading is the degree of a differential form, the differential is provided by the exterior differential and the wedge product gives the multiplication. For the singular cochain complex the grading is given by the degree of a singular cochain, the differential is the usual differential δ and multiplication is given by the cup-product.

Note that by introducing a zero differential we can turn any graded algebra into a differential graded algebra, for example a given cohomology ring.

An *augmentation* of a differential graded algebra is a map $\epsilon : A \to k$. If A has an augmentation it is called *augmented*. A choice of augmentation is similar to a choice of basepoint for a topological space, as we will see later. For an augmented differential graded algebra we can introduce the *augmentation ideal* $IA := \ker \epsilon : A \to k$. Furthermore let the *indecomposables* be $QA := IA/(IA \cdot IA)$.

Note that a differential graded algebra has a cohomology ring and we can talk about maps which induce an isomorphism on cohomology, so-called *quasi-isomorphism*. A rational equivalence between nilpotent spaces is precisely a map inducing a quasi-isomorphism on the rational singular cochains.

There is one more condition on a graded algebra we can hope to meet. A differential graded algebra is *graded commutative* or just *commutative* if $xy = (-1)^{\deg x \deg y} yx$ for all $x, y \in A$. We refer to a commutative differential graded algebra as a *cdga*, and denote the category of cdga's by $\mathscr{D}$. The category of augmented cdgas will be denoted $\mathscr{D}_0$.

Note that the de Rham complex of a differentiable manifold is a cdga, while the singular cochains are not. The de Rham complex is now our candidate to extract the algebraic analogue of a rational space. However, it has two drawbacks: The de Rham complex only exists on differentiable manifolds and its coefficients live in $\mathbb{R}$, instead of $\mathbb{Q}$. While the choice of coefficients in $\mathbb{R}$ or $\mathbb{Q}$ does not make a difference for singular cohomology there is no “real space” analogous to a rational space. These problems turn out to be technical. They can be overcome with the appropriate formalism. We will construct a contravariant functor $\tilde{A} : \mathbf{Top} \to \mathscr{D}$ such that $\tilde{A}X$ maps to C^*X by a map inducing an isomorphism on rational cohomology. Until we develop this formalism we consider the de Rham functor from differentiable manifolds to cdga's.

3.2 Minimal models

We already know that the de Rham complex is in general too large to be useful and the cohomology ring does not carry enough information. So we look for a class of cdga's that one can work with and that is richer than the cohomology ring.

Minimal models are cdga's which are free and can be obtained in a step by step process. Let us get some terminology in place. We say a cdga B is *connected* if $\eta : k \cong B^0$. We define $B(n)$ to be the subalgebra generated by $B^i, 0 \leq i \leq n$ and dB^n. Also, let

$B(-1)$ be $\eta(k)$. Now define $B(n,0)$ to be $B(n-1)$ and $B(n,p)$ to be the sub algebra generated by $B(n,p-1)$ and all the elements $x \in B^n$ with $dx \in B(n,p-1)$.

Definition. A free connected cdga M is called *minimal* if it satisfies $M(n) = \bigcup_p M(n,p)$ for all n. A *minimal model for a cdga B* is a minimal model M together with a quasi-isomorphism $e : M \to B$. A *minimal model for a manifold X* is a minimal model for the de Rham complex of X.

This definition implies that the differential on a minimal model M is decomposable. In the simply connected case (that is, if M is connected and $M^1 = 0$) this is equivalent to the given definition. Note also that since a minimal algebra is connected it has a unique augmentation $\epsilon : M \to k$ that is the isomorphism on M^0 and kills everything else.

Example 3.2. Let us calculate the minimal model M for the n-sphere. The de Rham complex of S^n is generated by the volume form of degree n. So we need a generator x in degree n, with $dx = 0$. If n is odd, the obvious map induces an isomorphism and M is the exterior algebra on one generator. If n is even, x generates a polynomial algebra. Since M must be free the only way to kill x^2 is to introduce an element y of degree $2n-1$ with $dy = x$. This is all.

We will later show (6.4) that for reasonable spaces the generators in degree $n \geq 2$ of the minimal model correspond to a basis of $\pi_n \otimes \mathbb{Q}$. So we can read all the non-torsion homotopy of the sphere from its minimal model. The homotopy groups for an odd-dimensional sphere S^n are torsion except in dimension n. The homotopy groups for an even-dimensional sphere S^n are torsion except in dimensions n and $2n-1$.

A word about notation: We write the minimal model for S^{2k} as $\Lambda(x, y \mid dy = x^2)$. In general we write $\Lambda(x_1, \ldots, x_n \mid dx_1 = s_1, \ldots, dx_n = s_n)$ for the minimal model on generators x_i with $dx_i = s_i$, where we leave out the dx_i which are 0. We may also write $\Lambda(x, dx)$ for $\Lambda(x, y \mid dx = y)$.

Example 3.3. Consider the space $(S^3 \times S^2) \# (S^3 \times S^2)$. The de Rham cohomology of $S^3 \times S^2$ is generated by elements in degree three and two, with nonzero product. If we now take the connected sum we have generators a, x in dimension two, generators b, y in degree three and the only nontrivial-product is $a \wedge b = -x \wedge y$. We will now calculate the minimal model. We need elements a, x representing the classes in degree two with $da = dx = 0$. Similarly, in dimension three we need generators b, y with $db = dy = 0$. But since $d(a \wedge x) = 0$ we also need a generator m_1 in dimension three to kill this element on cohomology. We also need to get rid off x^2 and a^2, so we need to introduce m_2 and m_3. In degree five we have to take care of the elements $x \wedge b, a \wedge y$ which have zero differential. So we need generators n_1, n_2 in degree four to kill them. Note that $dx \wedge m_1 = x \wedge x \wedge a \neq 0$ and similarly for all the other products of x, y and m_i.

$$\begin{array}{|cccc}
 & & & \vdots \\
 & & n_2 & ay = dn_2 \\
 & & n_1 & xb = dn_1 \\
 & m_3 & a^2 = dm_3 & \\
 & m_2 & x^2 = dm_2 & \\
 & m_1 & xa = dm_1 & \\
x & y & & \\
a & b & & ab = -xy \\
\hline
\end{array}$$

The process should now be clear: In degree n we determine all products of degree n and for every linear combination p with $d : p \mapsto 0$ which does not represent an element in cohomology we add a generator q of degree $n-1$, with $d : q \mapsto p$. Note that in degree six we need to consider twenty-two different products. So the procedure is lengthy, but completely formal and we can calculate the rational homotopy groups of $(S^3 \times S^2) \# (S^3 \times S^2)$ with arbitrary precision. We have already shown the following.

$$\pi_n\left((S^3 \times S^2) \# (S^3 \times S^2)\right) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}^2 & \text{if } n = 2 \\ \mathbb{Q}^5 & \text{if } n = 3 \\ \mathbb{Q}^2 & \text{if } n = 4 \end{cases}$$

Example 3.4. Let us attempt to calculate the minimal model for $(S^1 \times S^2) \# (S^1 \times S^2)$. Similarly to the previous example, the de Rham cohomology of $S^1 \times S^2$ is generated by elements in dimension two and one, with nonzero product. If we now take the connected sum we have generators a, x in dimension one, generators b, y in dimension two and the only nontrivial-product is $a \wedge b = -x \wedge y$. So we start the minimal model with elements a, x in dimension one with $da = dx = 0$. Again we need to kill the product $a \wedge x$, so introduce k_0 with $dk_0 = a \wedge x$. But now k_0 has degree one. So we have elements $a \wedge k_0$ and $x \wedge k_0$ in degree two. We find $d(a \wedge k_0) = 0 \wedge k_0 - a \wedge (a \wedge x) = 0$. So we need a new generator k_a in degree one with $dk_a = a \wedge k_0$. Similarly we need $dk_x = x \wedge k_0$. And again we find that $d(a \wedge k_a) = 0$, so we need another generator to k_{2a} to kill this product. This gives two infinite families of generators, k_{na} and k_{nx} with

$k_{0a} = k_{0x} = k_0$ and for $n \geq 1$ we have:

$$d : k_{na} \mapsto a \wedge k_{(n-1)a}$$
$$d : k_{nx} \mapsto x \wedge k_{(n-1)x}$$

To show that these are all the generators of degree one, it is enough to look at the following products and their differentials, and after some calculations we find they are indeed linearly independent.

$$\begin{array}{lcll}
x \wedge k_{na} & \overset{d}{\mapsto} & x \wedge a \wedge k_{(n-1)a} & \\
a \wedge k_{nx} & \overset{d}{\mapsto} & a \wedge x \wedge k_{(n-1)x} & \\
k_0 \wedge k_{nx} & \overset{d}{\mapsto} & a \wedge x \wedge k_{nx} - k_0 \wedge x \wedge k_{(n-1)x} & \\
k_0 \wedge k_{na} & \overset{d}{\mapsto} & a \wedge x \wedge k_{na} - k_0 \wedge a \wedge k_{(n-1)a} & \\
k_{na} \wedge k_{ma} & \overset{d}{\mapsto} & a \wedge k_{(n-1)a} \wedge k_{ma} - k_{na} \wedge a \wedge k_{(m-1)a} & \text{if } n, m \geq 1, n \neq m \\
k_{nx} \wedge k_{mx} & \overset{d}{\mapsto} & x \wedge k_{(n-1)x} \wedge k_{mx} - k_{nx} \wedge x \wedge k_{(m-1)x} & \text{if } n, m \geq 1, n \neq m \\
k_{nx} \wedge k_{ma} & \overset{d}{\mapsto} & x \wedge k_{(n-1)x} \wedge k_{ma} - k_{nx} \wedge a \wedge k_{(m-1)a} & \text{if } n, m \geq 1
\end{array}$$

To find all the generators in degree two we have to look at products in degree three. There will be a variety of those and it is already evident that the minimal model of $(S^1 \times S^2) \# (S^1 \times S^2)$ is a rather nasty object. We won't make any attempts to calculate more of it.

This example illustrates that rational homotopy theory is far less tractable if our spaces have unpleasant fundamental groups. If we recall the conditions for the correspondence between minimal models and rational spaces, we find that the space be nilpotent and this implies that the fundamental group be *nilpotent*, i.e. the lower central series, constructed by repeatedly taking commutator subgroups, terminates after a finite number of steps. In this case $\pi_1((S^1 \times S^2) \# (S^1 \times S^2)) \cong \mathbb{Z} * \mathbb{Z}$, which is not nilpotent. So the minimal model for $(S^1 \times S^2) \# (S^1 \times S^2)$ is not only harder to calculate but also less useful then that for, say, $(S^3 \times S^2) \# (S^3 \times S^2)$.

Example 3.5. Let us finally look at a space whose minimal model is somewhat different from previous ones. We will look at this difference in some detail in the last chapter of this essay. For now consider the following subgroups of $SL_3(\mathbb{R})$.

$$N := \{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}\} \qquad \Gamma := \{\begin{pmatrix} 1 & k & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \mid k, l, m \in \mathbb{Z}\}$$

Let $N_\Gamma := N/\Gamma$. We find three invariant differentials dx, dz and $dy - xdz$ which generate the de Rham cohomology of N_Γ.

$$H^n(N_\Gamma, \mathbb{R}) = \begin{cases} \mathbb{R}^2 = \langle dx, dz \rangle & \text{if } n = 1 \\ \mathbb{R}^2 = \langle dx\, dy - x\, dx\, dz, dy\, dz \rangle & \text{if } n = 2 \\ \mathbb{R}^1 = \langle dx\, dy\, dz \rangle & \text{if } n = 3 \end{cases}$$

It is readily seen that we have a minimal model $\Lambda(a, b, c \mid dc = ab)$. Note that this is different from the minimal model for $(S^1 \times S^2) \# (S^1 \times S^2)$, which has the same cohomology as N_Γ.

Now that we have some idea how minimal models are calculated, we can go back to the theory. We only need one condition on a cdga to guarantee the existence of a minimal model. We say a cdga B is *homologically connected* if $H^0B \cong k$.

Theorem 3.1. *Every homologically connected cdga B has a minimal model.*

Proof. We will "just do it" and build the minimal model step by step. We will construct $f_{n,p} : M(n,p) \to B$, where the $M(n,p)$ are as in our definition of a minimal model and $f_{n,p}$ induces an isomorphism on cohomology in degree up to $n-1$ and an injection in degree n. Let $f_{1,0} : M(1,0) \to B$ be $\eta : k \to B$. Now assume we have defined $f_{n,p}$ for $n \geq 1, p \geq 0$. Pick cocycles $b_1, \ldots, b_k \in B^n$ representing a basis for the cokernel of $H^n f_{n,p}$. Similarly, pick representatives $x_1, \ldots, x_j \in M(n, p-1)^{n+1}$ for a basis of the kernel of $H^{n+1} f_{n,p}$. We know $f_{n,p} x_i = dc_i$ for some $c_i \in B^{n+1}$. Now define $M_{n,p+1}$ by adding generators $m_1, \ldots, m_k$ in degree n with $dm_i = 0$ and generators $n_1, \ldots, n_j$ in degree n with $dn_i = x_i$. Define $f_{n,p+1}$ to be $f_{n,p}$ on $M(n,p)$ and $f_{n,p+1} : m_i \mapsto b_i$, $f_{n,p+1} : n_i \mapsto c_i$. We need to check that $f_{n,p+1}$ still induces isomorphisms on $H^{\leq n-1}$ and an injection on H^n. But we changed nothing in degrees smaller than n and the cohomology elements we added in degree n map to a basis for the cokernel, so cannot add to the kernel in degree n.

Now let $M(n+1, 0) = \bigcup_p M(n,p)$ and define $f_{n+1,0}$ by $f_{n+1,0}|_{M(n,p)} = f_{n,p}$. We claim that by construction $H^n f_{n+1,0}$ is an isomorphism and $H^{n+1} f_{n+1,0}$ is an injection. Indeed, suppose x represents an element in the kernel of $H^{n+1} f_{n+1,0}$. Then $x \in M(n,p)$ for some p and $H^{n+1} f_{n,p} x = 0$. But then $x = dn$ for some $n \in M(n, p+1)$. For the isomorphism observe first that injectivity holds for all $H^n f_{n,p}$ by induction and thus for $H^n f_{n+1,0}$. Surjectivity holds for $H^n f_{n,1}$ and from then on, by construction. Finally let $M = \bigcup_n M(n,0)$ and extend the $f_{n,0}$ in the obvious way to a map $f : M \to B$. It is immediate that $H^* f$ is an isomorphism. □

3.3 Homotopy Theory in $\mathscr{D}$

We need to develop some more theory in $\mathscr{D}$ to understand more about minimal models. This theory begins with the observation that $\mathscr{D}$ admits homotopy: A *homotopy* between two maps $f, g : A \to B$ is a map $H : A \to \nabla(1,*) \otimes B$ with $(\partial_0 \otimes \mathbf{1}) \circ H = f$ and $(\partial_1 \otimes \mathbf{1}) \circ H = g$. Here $\nabla(1,*)$ is the free algebra on generators d, dt with differential $d : t \mapsto dt$ and we define maps $\partial_0 t \mapsto 0, \partial_1 t \mapsto 1$. (This notation will be generalised in chapter 5.) Note that this definition is somehow dual to the definition of a homotopy in **Top**, as is to expected since we want to connect the two categories by contravariant functors, like the de Rham functor.

Similarly, there is a homotopy theory for augmented cdgas. Define the homotopy as above, replacing $\otimes$ by $\tilde{\otimes}$ defined as follows:

$$\nabla(1,*) \mathbin{\tilde{\otimes}} B := k \otimes k \oplus \nabla(1,*) \otimes IB$$

This phenomenon actually lives in a bigger context. It is crucial to observe that $\mathscr{D}$ is a *model category*. So instead of proving specific results about homotopy theory in $\mathscr{D}$ we will derive them from homotopy theory in model categories, introduced in the next chapter. Now we will just prove the following lemma:

Lemma 3.2. *Let $f, g : A \to B$ be homotopic in $\mathscr{D}$. Then $H^* f = H^* g$. The same result holds in $\mathscr{D}_0$.*

Proof. Consider a homotopy $J : A \to \nabla(1,*) \otimes B$ between f and g. It is enough to show that $\partial_0 \otimes \mathbf{1}$ and $\partial_1 \otimes \mathbf{1}$ are equal on cohomology. But $H^*\nabla(1,*) = k$, as can easily be seen, so the result follows from the algebraic Künneth theorem. The same proof applies in $\mathscr{D}_0$. □

This (together with basic properties established in the next chapter) implies that homotopy equivalences are quasi-isomorphisms. Hence minimal models are homotopy invariant. Given a homotopy equivalence $f : A \to B$ any minimal model $e_A : M \to A$ is a minimal model for B by $f \circ e_A : M \to B$. However, we still haven't shown whether minimal models are unique.

4 Model Categories

The notion of a model category was introduced by Quillen in the 60's and has been developed since then. Model categories like **Top** or $\mathscr{D}$ are the natural habitat for homotopies, as we shall see in the following. We will only have time to establish the language here. For a very readable introduction see [6], for a more complete account see [12].

4.1 Definition

We will need the following terminology. Given a commutative diagram of solid arrows

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\
V \arrow[r] \arrow[ur, dashed, "q"] & P
\end{tikzcd}$$

we say that i has the *left lifting property* with respect to p if the dotted arrow q exists. We also say p has the *right lifting property* with respect to i.

Definition. A *Model Category* is a category $\mathscr{C}$ with three distinguished kinds of morphisms, *weak equivalences*, written as $\xrightarrow{\sim}$, *fibrations* ($\twoheadrightarrow$), and *cofibrations* ($\hookrightarrow$) satisfying the following axioms. A fibration or cofibration that is also a weak equivalence is called *trivial*.

1. $\mathscr{C}$ has all finite limits and colimits.
2. If any two of f, g and fg are weak equivalences, then so is the third.
3. Every retract of a weak equivalence, fibration or cofibration is itself a weak equivalence, fibration or cofibration respectively.

4. Cofibrations have the left lifting property with respect to trivial fibrations. Fibrations have the right lifting property with respect to trivial cofibrations.

5. Any morphism f factors as

    a) $f = q \circ i$ where i is a trivial cofibration and q is a fibration,

    b) $f = p \circ j$ where j is a cofibration and p is a trivial fibration.

Since model categories have finite limits and colimits, they have an initial object 0 and a terminal object 1. We call an object X *fibrant* if $X \to 1$ is a fibration and *cofibrant* if $0 \to X$ is a cofibration.

One can prove that fibrations and cofibrations determine each other via the lifting properties, i. e. given a class of fibrations we can define a map to be a cofibration precisely if it has the left lifting property with respect to trivial fibrations.

The first natural example of a model category is **Top**, the category of topological spaces. In **Top** weak equivalences are precisely that (i. e. maps inducing isomorphisms on all homotopy groups), fibrations are Serre-fibrations and cofibrations can then be defined via the lifting property. We already mentioned that $\mathscr{D}$ is also a model category and in the next chapter we will meet the very important model category of simplicial sets.

4.2 Some Homotopical Algebra

There are two ways of defining a homotopy in a model category. They correspond to the dual ways in which we defined homotopies in **Top** and in $\mathscr{D}$.

Definition. Let $f, g : A \to B$ be morphisms in $\mathscr{D}$. A *cylinder object* for A is an object A' such that the natural map $A \amalg A \to A$ factors as $A \amalg A \hookrightarrow A' \xrightarrow{\sim} A$. We write the cofibration as $i_0 + i_1$. A *left homotopy* between f and g is a map $H : A' \to B$ for any cylinder object A' for A, such that $Hi_0 = f$ and $Hi_1 = g$. We write $f \overset{l}{\sim} g$ if there is a left homotopy between f and g.

A *path object* for B is an object B' such that the diagonal map $B \to B \times B$ factors as $B \xrightarrow{\sim} B' \twoheadrightarrow B \times B$. We write the fibration as (p_0, p_1). A *right homotopy* is a map $K : A \to B'$ for any path object B for B, such that $p_0 K = f$ and $p_1 K = g$. We write $f \overset{r}{\sim} g$ if there is a right homotopy between f and g.

The usual homotopies in **Top** are left homotopies with cylinder object $X \times I$ for X. We will see that the homotopy we defined in $\mathscr{D}$ is a right homotopy. However, we must be careful about the terminology. The existence of a right homotopy in the model category sense does not guarantee a homotopy through any specific path object, so in particular not through the path object we picked earlier.

Left and right homotopies are in general not equivalence relations since transitivity can fail. However, between "nice" objects there is a good homotopy theory and frequently both notions of homotopy coincide. We will now quote a number of standard results about homotopies in model categories. They are proven for example in [12]. Note that left and right homotopy are dual, so all theorems about left homotopy apply to right homotopy when dualised.

Lemma 4.1. *Let $\mathscr{C}$ be a model category, $A, B, C \in ob\,\mathscr{C}$ and $f, g : A \to B$.*

1. *If $f \overset{l}{\sim} g$ then for $h : B \to C$ we have $hf \overset{l}{\sim} hg$. If A is cofibrant then we also have $fh \overset{l}{\sim} gh$ for $h : C \to A$.*
2. *Left and right homotopy are always reflexive and symmetric. If A is cofibrant, $\overset{l}{\sim}$ is an equivalence relation on $\mathscr{C}(A, B)$.*
3. *Let A be cofibrant and $f : B \to C$ a weak equivalence between fibrant objects. Then there is an isomorphism $f_* : \mathscr{C}(A, B)/\overset{l}{\sim} \to \mathscr{C}(A, C)/\overset{l}{\sim}$.*
4. *Let A be cofibrant and B fibrant. Then $f \overset{l}{\sim} g$ if and only if $f \overset{r}{\sim} g$. Furthermore, if f and g are homotopic we can pick the homotopy through any cylinder object or path object.*
5. *Let A and B be fibrant and cofibrant. Then a weak equivalence $f : A \to B$ is precisely a homotopy equivalence.*

Given a model category $\mathscr{C}$ we want to consider it "up to homotopy". To do this we form its *homotopy category*. However, there are various ways to do this.

The obvious approach is to replace morphisms by equivalence classes of homotopic morphisms. But we can only do this for morphisms between fibrant cofibrant objects. By the factorisation axiom we can always find a fibrant replacement QX and a cofibrant replacement RX for an element X of $\mathscr{C}$. For every element X of $\mathscr{C}$ fix a fibrant cofibrant replacement RQX once and for all. Then we can define a category $Ho\mathscr{C}$ and a functor $\gamma_{\mathscr{C}} : \mathscr{C} \to Ho\mathscr{C}$ as follows. Let $\gamma_{\mathscr{C}}$, abbreviated γ, be the identity on objects. However, $Ho\mathscr{C}[\gamma X, \gamma Y] := \mathscr{C}[RQX, RQY]/\sim$, where $\sim$ is the homotopy relation. We will often abbreviate this set of homotopy classes of maps to $[X, Y]$. It is not hard to show that there is a map $\tilde{f}$ making the following diagram commute up to homotopy.

$$\begin{tikzcd}
RQX \arrow[r, "\tilde{f}"] \arrow[d] & RQY \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Now define $\gamma f := [\tilde{f}]$.

We can also denote by W the class of weak equivalences in $\mathscr{C}$ and define the category $W^{-1}\mathscr{C}$ to be the localisation of $\mathscr{C}$ with respect to weak equivalences. That is, there is an inclusion $\delta : \mathscr{C} \to W^{-1}\mathscr{C}$ that takes weak equivalences to isomorphisms and for any functor $F : \mathscr{C} \to \mathscr{D}$ which takes weak equialences to isomorphisms, there is a unique factorisation $\mathscr{C} \xrightarrow{\delta} W^{-1}\mathscr{C} \to D$. This is achieved by formally inverting all weak equivalences. (We will not describe the formalism needed to make this work.)

We will be interested in homotopy categories in chapter 6. For now we will conclude by stating the main result about homotopy categories.

Theorem 4.2. *The categories $W^{-1}\mathscr{C}$ and $Ho\mathscr{C}$ are equivalent.*

Remark. We are naturally also interested in the question how we can lift a functor $F : \mathscr{C} \to \mathscr{D}$ to a functor $Ho\mathscr{C} \to Ho\mathscr{D}$. Ideally we would like to factor $\gamma_{\mathscr{D}} \circ F$ as $HoF \circ \gamma_{\mathscr{C}}$. While this is occasionally possible, there is a more general answer to our question, the theory of *derived functors*. Introductions form different viewpoints can be found in [12] and [6].

4.3 $\mathscr{D}$ as a model category

It is proven in §4 of [2] that $\mathscr{D}$ is a model category. This allows us to prove the uniqueness of minimal models in a very natural way.

Theorem 4.3. *Call a map in $\mathscr{D}$ a weak equivalence if it is a quasi-isomorphism, a fibration if it is onto, and a cofibration if it has the left lifting property with respect to trivial fibrations. With these definitions $\mathscr{D}$ is a model category.*

One can show that whenever $\mathscr{C}$ is a model category, so is $\mathscr{C}/B$. By substituting $\mathscr{D}$ for $\mathscr{C}$ and k for B we find:

Corollary 4.4. *$\mathscr{D}_0$ is a model category.*

Note that $\nabla(1,*)\otimes X$ is a path object for X in $\mathscr{D}$. $X \to \nabla(1,*)\otimes X$ is a weak equivalence by the Künneth theorem and $(\partial_0, \partial_1) : \nabla(1,*)\otimes X \to X\times X$ is onto since $t_0\otimes x + t_1\otimes y \mapsto (x, y)$. So the homotopy we defined earlier is a right homotopy.

We observe that every object of $\mathscr{D}$ is fibrant, while not every object is cofibrant. It turns out that minimal models are cofibrant but before we can prove this, we will introduce some examples of cofibrations. Neither of the two following lemmas is hard to prove and both are essential for the proof of Theorem 4.3.

Lemma 4.5. *The algebras $\Lambda(x)$ and $\Lambda(y, dy)$ are cofibrant. The map $\theta : \Lambda(x) \to \Lambda(y, dy)$ defined by $x \mapsto dy$ is a cofibration.*

Lemma 4.6. *Cofibrations are closed under push-out, countable composition and arbitrary tensor products.*

Lemma 4.7. *Any minimal algebra M is a cofibrant object in $\mathscr{D}$.*

Proof. We will show that every inclusion $M(n, p-1) \hookrightarrow M(n, p)$ is a cofibration. The result then follows by the previous lemma about closure properties of cofibrations. The main point is to pick an appropriate set of generators for M. Let $G(1, 0) = k$. Then for each $n, p \geq 1$ extend $G(n, p-1)$ to $G(n, p)$ by adding elements in X^n such that $G(n, p)$ projects to a basis for $X(n, p)^n / X(n, 0)^n$. Then $\bigcup_p G(n, p)$ projects to a basis for the indecomposables $QX^n = X^n / X(n, 0)^n$. Thus M is freely generated by $\bigcup_{n,p} G(n, p)$. Now $X(n, p)$ is freely generated by $G(n, p)$ and the differential determines the following push-out diagram, where $\deg x = n + 1$ and $\theta : x \mapsto dy$.

$$\begin{tikzcd}
\otimes \Lambda(x) \arrow[r] \arrow[d, "\theta"'] & X(n, p) \arrow[d] \\
\otimes \Lambda(y, dy) \arrow[r] & X(n, p+1)
\end{tikzcd} \tag{4.1}$$

Since the inclusion on the left is a cofibration and push-outs of cofibrations are cofibrations the lemma follows. □

Lemma 4.8. *A weak equivalence between minimal models is an isomorphism.*

Proof. Using the unique augmentation of minimal models, we work in $\mathscr{D}_0$. A weak equivalence between minimal models is a homotopy equivalence by Lemma 4.1. We claim that homotopic maps $f, g : M \to N$ induce the same map $HQM \to HQN$. Then the result follows, since the differential on a minimal model is decomposable, so that we have $QM \cong HQM \cong HQN \cong QN$, and a morphism is determined by its values on indecomposables. To prove our claim observe that as in the proof of (3.2) by Künneth's theorem $H\partial_0 = H\partial_1 : H(\nabla(1,*) \otimes QN) \to H(\nabla(0,*) \otimes QN)$. But now looking back at the definitions we find

$$HQ\partial_0 = HQ\partial_1 : HQ(\nabla(1,*) \tilde{\otimes} N) = HQ(\nabla(0,*) \tilde{\otimes} N)$$

Replacing our homotopy $J : M \to \nabla(1,*) \tilde{\otimes} N$ by HQJ the result follows. □

Theorem 4.9. *The minimal model of a cdga is unique up to isomorphism.*

Proof. Suppose we have a cdga B and two minimal models $f : M \xrightarrow{\sim} B$ and $f' : M' \xrightarrow{\sim} B$. Since every object in $\mathscr{D}$ is fibrant and minimal models are cofibrant, homotopy with any given path object forms an equivalence relation in $\mathscr{D}(M, B)$ and $\mathscr{D}(M', B)$. Denote the set of equivalence classes by $[M, B]$ and $[M', B]$. Composition with f' induces an isomorphism from $[M, M']$ to $[M, B]$. So there is $g : M \to M'$ such that $f' \circ g \sim f$ and we can choose the homotopy to go via the $\nabla(1,*) \otimes B$, all by (4.1). Thus $H^*f = H^*f' \circ H^*g$ by (3.2), and g is a weak equivalence between minimal models and thus an isomorphism. □

In the model category context a minimal model is a cofibrant replacement for a (homologically connected) object in $\mathscr{D}$, i. e. our construction of the minimal model for B is a factorisation of the map $* \to B$. Similarly we can find minimal factorisations for more general maps and this is one ingredient in the proof of the following theorem, an algebraic Whitehead theorem, that we will only state here. The proof can be found in [2].

Theorem 4.10. *Let $f : A \to B$ be a map of connected cofibrant augmented algebras. Then $H^i f$ is an isomorphism for $i \leq n$ and an injection for $i = n+1$ if and only if $H^i Qf$ is an isomorphism on $i \leq n$ and an injection for $i = n+1$.*

5 The Fundamental Adjunction

Our next step towards the identification of homotopy classes of rational spaces and minimal model is the construction of a contravariant adjunction between topological spaces and differential graded algebras. The motivating idea is that we can generalise and rationalise the de Rham complex by considering the algebra of polynomial forms on simplices instead.

Before we embark on this, we will swap categories. It is more convenient for the task at hand not to work with topological spaces but rather in the category $\mathscr{S}$ of simplicial sets, which will be introduced below. This is legitimate since the singular functor $S : \mathbf{Top} \to \mathscr{S}$ has a left adjoint, the geometric realisation functor $|\,| : \mathscr{S} \to \mathbf{Top}$, and these two induce an equivalence of the homotopy categories. (Both $\mathscr{S}$ and **Top** are model categories.) Thus homotopy theory in $\mathscr{S}$ and **Top** is almost identical and we will freely make use of this fact.

5.1 Interlude: Simplicial Sets

Unfortunately, we do not have time to develop the theory of simplicial sets in any detail, instead we refer the reader to [14] for a down-to-earth or [8] for a more modern exposition of the subject. Both texts treat geometric realisation and the equivalence of categories.

Let us at least establish the terminology:

Definition. A *Simplicial Set* is a set $K = \coprod_{p \geq 0} K_p$ with maps $\partial_i^K : K_p \to K_{p-1}$ and $s_i^K : K_p \to K_{p+1}$, usually abbreviated ∂_i, s_i, such that

1. $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$
2. $s_i s_j = s_{j+1} s_j$ for $i \leq j$
3. $\partial_i s_j = s_{j-1} \partial_1$ for $i < j$

   $\partial_i s_i = \partial_{i+1} s_i = id.$

$\partial_i s_j = s_j \partial_{i-1}$ if $i > j + 1$

The elements K_p are called the *p-simplices*. The maps ∂_i and s_j are called *face* and *degeneracy operations* respectively. A simplex of the form $s_i x$ is called *degenerate*. A simplex not of this form is called *non-degenerate*. The simplices $\partial_i k$ for $k \in K_p$ are called the faces of k. A *simplicial map* $f : K \to L$ is a collection of maps $f_n : K_n \to L_n$, that is compatible with the simplicial structure in the following sense: $s^L_n f_n = f_{n+1} s^K_n$ and $\partial^L_n f_n = f_{n-1} \partial^K_n$. We define the *n-th skeleton* $\mathrm{sk}_n X$ of a simplicial set to be the subcomplex generated by simplices of degree $\leq n$. Then we have $X = \bigcup \mathrm{sk}_n X$.

Simplicial complexes and singular simplices are good examples of simplicial sets to keep in mind.

There is also a categorical definition of a simplicial set. First we define the category $\mathbb{\Delta}$ as follows. For every $n \in \mathbb{N}$ let ob $\mathbb{\Delta}$ contain the object $[n]$, a totally ordered set of size $n + 1$. The morphisms of $\mathbb{\Delta}$ are just order-preserving set-maps. Then a simplicial set is a contravariant functor $S : \mathbb{\Delta} \to \mathbf{Set}$. A simplicial map is a natural transformation between simplicial sets. It is straightforward to check that the definitions coincide. They also allow to easily generalise simplicial sets to *simplicial objects* in arbitrary categories.

Homotopy is not in general well-behaved in $\mathscr{S}$, however, we can define the homotopy groups of an arbitrary simplicial set to be the (topological) homotopy groups of its realisation. Correspondingly a *weak equivalence* in $\mathscr{S}$ is a map $f : A \to B$ such that the realisation $|f| : |A| \to |B|$ is a weak equivalence. A morphism of $\mathscr{S}$ is a cofibration if it is an inclusion. Fibrations can then be defined via the lifting property. It is an important theorem in the development of the theory of simplicial sets that the realisation of a fibration in $\mathscr{S}$ is a Serre fibration. Fibrations in $\mathscr{S}$ are called *Kan fibrations* and fibrant objects *Kan complexes*.

Let us briefly mention the usual definition of Kan fibrations. The *standard n-simplex* Δ^n has m-dimensional simplices $\mathbb{\Delta}([m], [n])$, i. e. non-decreasing sequences of numbers from $[n]$. It has face and degeneracy operators d_i and s_i defined by omitting respectively repeating the i-th element in the sequence. The *k-th horn* $\Lambda^n_k$is the subcomplex of Δ^n generated by all faces of the single n-simplex except the k-th. Now a map $p : X \to$ in $\mathscr{S}$ is a Kan fibration if and only if it has the right lifting property with respect to all inclusions $\Lambda^n_k \hookrightarrow \Delta^n$. (The slogan is "Every horn can be filled.").

Theorem 5.1. *With weak equivalences, cofibrations and fibrations defined as above, $\mathscr{S}$ is a model category.*

Convention. The rest of this essay is written simplicially: Whenever we say X is a *space* we mean $X \in \operatorname{ob} \mathscr{S}$, when we say f is a *map* between spaces we mean $f \in \operatorname{mor} \mathscr{S}$.

5.2 The Double Complex $\nabla(*, *)$

For each n we construct the differential graded algebra $\nabla(n, *)$, defined to be the free commutative algebra over $\mathbb{Q}$ with generators $t_1, \ldots, t_n, dt_1, \ldots, dt_n$, where the degree of t_i is 0 and the degree of dt_i is 1 and we define the differential $d : t_i \mapsto dt_i$. Then $\nabla(n, p)$ is the set of elements of degree p.

Next we introduce face and degeneracy maps. Let $s_i : \nabla(n, *) \to \nabla(n + 1, *)$ and $\partial_i : \nabla(n, *) \to \nabla(n - 1, *)$ be morphisms in $\mathscr{D}$ such that.

$$s_i t_k = \begin{cases} t_{k+1} & \text{if } i < k \\ t_k + t_{k+1} & \text{if } i = k \\ t_k & \text{if } i > k \end{cases} \qquad \partial_i t_k = \begin{cases} t_{k-1} & \text{if } i < k \\ 0 & \text{if } i = k \\ t_k & \text{if } i > k \end{cases}$$

Note that the t_k on the right hand side and on the left hand side of these equations live in different $\nabla(n, *)$ and are different elements.

A straightforward calculation shows that this makes $\nabla(*, p$ into an object of $\mathscr{S}$. To prove, for example, $\partial_i \partial_j = \partial_{j-1} \partial_i$ (assuming $i < j$) it is enough to check the following:

$$\partial_i \partial_j t_k = \left\{ \begin{array}{ll} t_{k-2} & \text{if } j < k \\ 0 & \text{if } j = k \\ t_{k-1} & \text{if } i < k, j > k \\ 0 & \text{if } i = k \\ t_k & \text{if } i > k \end{array} \right\} = \partial_{j-1} \partial_i t_k$$

By definition s_i and ∂_i commute with differential and multiplication. We now observe that $\nabla(*, *)$ lives in $\mathscr{S}$ and in $\mathscr{D}$. This allows the crucial idea that if we consider the set of maps from an object X to $\nabla(*, *)$ in one category there is structure “left over” in the other category: If we let X be a space then $A : X \mapsto \mathscr{S}(X, \nabla(*, *))$ is a functor from $\mathscr{S}$ to $\mathscr{D}$. Similarly we have $F : B \mapsto \mathscr{D}(B, \nabla(*, *))$ going from $\mathscr{D}$ to $\mathscr{S}$ and it should be no surprise that the two functors are adjoint. Note that the functor $\tilde{A}$ promised earlier is just the composition AS.

Theorem 5.2. *There is a contravariant adjunction* $A : \mathscr{S} \rightleftarrows \mathscr{D} : F$.

Proof. Let $A^pX = \mathscr{S}(X, \nabla(*, p))$ and define $A : X \mapsto A^*X$. We must show AX is a cdga. The obvious choices for differential, unit and multiplication do the job. Let $d_{AX}(f) = d_{\nabla(*,*)} \circ f$. Also, let $\eta_{AX}(r) : x \mapsto \eta_{\nabla(*,*)}(r)$ and $\mu_{AX}(f, g) : x \mapsto \mu_{\nabla(*,*)}(f(x), g(x))$. These satisfy the axioms for a cdga since their counterparts in $\nabla(*, *)$ do. Similarly let $F^nB = \mathscr{D}(B, \nabla(n, *))$ and $F : B \mapsto F^*B$. Again we can construct the necessary maps in FB from those in $\nabla(*, *)$ by composition, making FB into a simplicial set as promised.

Once we have defined F and A it is not hard to show they are adjoint. We have to show that there is a natural bijection:

$$\mathscr{S}(X, FB) \cong \mathscr{D}(B, AX)$$

Given a simplicial map $f : X \to FB$ we want to define a map $f^\# : B \to FX$ in $\mathscr{D}$. Observe that for $x \in X^n$, we have $f(x) : B \to \nabla(n, *)$, and for $b \in B^p$ we have $f(x)(b) \in \nabla(n, p)$. So we define $f^\#(b)(x) = f(x)(b) \in \nabla(n, p)$. This clearly gives a natural bijection. $\square$

Corollary 5.3. *There is a contravariant adjunction between* **Top** *and* $\mathscr{D}$.

Observe that a choice of basepoint $x : * \to X$ becomes an augmentation $Ax : AX \to \nabla(0, 0) = k$ under this functor. So we get an adjunction of augmented cdgas $\mathscr{D}_0$ and pointed simplicial sets $\mathscr{S}_0$ or topological spaces $\mathbf{Top}_0$.

We will now state two very important result about the functors of this adjunction F. They are technical results proven in [2].

Lemma 5.4. *The functors* F *and* A *take cofibrations to fibrations.*

Lemma 5.5. *The functor* $F : \mathscr{D} \to \mathscr{S}$ *preserves homotopies.*

5.3 Cohomology of AX

We have constructed A without checking what kind of information about X it contains. We will now sketch a proof that A gives the same cohomology as the (*normalised*) singular cochain functor C^*. (The normalised chain complex is obtained by dividing out all degenerate simplices.)

First we need to construct a map from AX to C^*X which will induce the isomorphism on cohomology. We will obtain this map ρ as formal integration of polynomial forms. Given an element $s = s(t_1, \dots, t_p)\, dt_1 \dots dt_p \in \nabla(p, p)$ and a simplex $\nabla^p = \{t_i \mid 0 \leq i \leq p, 0 \leq t_i, \sum t_1 = 1\}$ we can define

$$\int s = \begin{cases} \int_{\nabla^p} s(t_1, \dots, t_p)\, dt_1 \dots dt_p & \text{if } p > 0 \\ s & \text{if } p = 0 \end{cases}$$

as a real integral. Since the result is a polynomial, we can define $\int s$ over any field of characteristic 0, in particular over the rationals.

We can define a *total differential* $\partial = \sum (-1)^i \partial_i : \nabla(n, p) \to \nabla(n-1, p)$. This allows us to state Stokes' theorem (in the version appropriate for us).

Proposition 5.6. *Let $s \in \nabla(p, p-1)$. Then $\int ds = \int \partial s$.*

Next we define $\rho : A^*X \to C^*X$. Let $w \in A^pX$, so $w : X \to \nabla(*, p)$. Then for $x \in X^n$ we have $\rho(w)(x) = \int wx$.

Proposition 5.7. *ρ is a chain map into C^*X respecting the unit.*

Proof. Note first that $\rho\eta = \eta$ by the definition of $\int$ on degree 0. Also $\rho(w)(sx) = \int swx = 0$ since $wx \in \nabla(p-1, p) = 0$, so ρ really is a map into normalised cochains.

Finally consider $\rho(dw)(x) = \int dw(x) = \int \partial w(x)$ by Stokes' theorem. $\int \partial w(x) = \int w(\partial x) = d \int wx = d\rho(w)(x)$ by the definition of the differential on cochains. Thus ρ is a chain map. □

Theorem 5.8. *Let X be an object of $\mathscr{S}$. Then $H(AX) \cong H(C^*X)$ as rings.*

Sketch of proof. Note first that ρ is not multiplicative, however it is an easy consequence of the fact the ρ is *strongly homotopy multiplicative* (see Idea of proof for Lemma 6.9) that ρ is multiplicative on cohomology.

To show that ρ is a group isomorphism one can use an induction along the n-skeleton of X. We need four ingredients.

- $HA(\Delta^n) \cong HC^*(\Delta^n)$

- There is an isomorphism $HA(\amalg X) \cong \prod HA(X)$.
- Given a push-out square in $\mathscr{S}$ there is an associated long exact Mayer-Vietoris sequence.
- For a sequence $X(0) \subset X(1) \subset \cdots$ and an element $X = \bigcup X(n)$ in $\mathscr{S}$ there is a short exact sequence

$$0 \to \varprojlim{}^1 H^{n-1}AX(i) \to H^n AX \to \varprojlim H^n AX \to 0$$

Note that there are results for singular cochains corresponding to the last three points and the constructions involved are natural with respect to ρ. So we just need to know that we can build any space X using these constructions. But this is precisely what we do in constructing the n-skeletons of a space. A complete proof along this lines can be found in §14 of [2]. □

Now we know enough about A to define the *minimal model for a space X* to be the minimal model $e_X : MX \xrightarrow{\sim} AX$. We will take this construction further in the next section.

5.4 The Minimal Model Functor and Homotopy Categories

We will consider the homotopy categories of $Ho\mathscr{S}$ and $Ho\mathscr{D}$, whose objects are those of $\mathscr{S}$ and $\mathscr{D}$ respectively and whose maps are maps between fibrant cofibrant replacements. Note that every object in $\mathscr{S}$ is cofibrant, so the fibrant cofibrant replacements are just the Kan complexes. Similarly every object in $\mathscr{D}$ is fibrant, so the replacements are the cofibrant algebras.

Recall that we defined the minimal model for X to be the minimal model $e_X : MX \xrightarrow{\sim} AX$. We can now define the functor $M : \mathscr{S} \to Ho\mathscr{D}$ to take a space X to a cofibrant replacement for AX, and to a minimal model of X if X is connected. We also send a morphism f to a morphism Mf making the following square commute up to homotopy.

$$\begin{tikzcd}
MX \arrow[r, "Mf"] \arrow[d, "e_x"'] & MY \arrow[d, "e_Y"] \\
AX \arrow[r, "Af"'] & AY
\end{tikzcd}$$

We know Mf exists and is unique since the weak equivalence e_Y induces an isomorphism between homotopy classes $[MX, Y]$ and $[MX, MY]$, under which $f \circ e_X$ corresponds to Mf. Now we would like to factor M through $\gamma_{\mathscr{S}} : \mathscr{S} \to Ho\mathscr{S}$, to obtain a functor on $Ho\mathscr{S}$. We can certainly do this if M takes fibrant cofibrant objects to fibrant cofibrant objects and homotopies to homotopies. But the first assertion follows from the definition of M and the fact that every object in $\mathscr{D}$ is fibrant. For the second assertion we look back at the way in which we just constructed Mf and realise that it is enough to have the following lemma.

Lemma 5.9. *Let $f \sim g : X \to Y$, where X, Y are fibrant. Then $Af_* = Ag_* : [M, AY] \to [M, AX]$ for any cofibrant M.*

For the proof we have to refer the reader to §8 of [2] once more. We will abuse notation and call the new functor M also. To show that we have a functor $F : Ho\mathscr{D} \to Ho\mathscr{S}$ we only need to recall that F takes cofibrants to fibrants by (5.4) and preserves homotopies by (5.5).

Theorem 5.10. *There are adjunctions $M : Ho\mathscr{S} \leftrightarrows Ho\mathscr{D} : F$ and $M : Ho\mathscr{S}_0 \leftrightarrows Ho\mathscr{D}_0 : F$.*

Proof. One can show this explicitly, but it also follows from Quillen's theory of derived functors. As noted in the remark at the end of section 4.2 a derived functor for a functor $G : \mathscr{C} \to \mathscr{D}$ is an approximation to G on the homotopy category. Since we obtained by M and F on the homotopy categories by factoring M and F they are (particularly nice) derived functors. Next it is a central result about derived functors, proved for example in [6], that an adjunction $G \dashv H$ induces an adjunction of derived functors on the homotopy category if G preserves cofibrations and H preserves fibrations. Taking into account that our functors are contravariant it suffices to show that they map cofibrations to fibrations. But M does this since A does and so this is (5.4). All the observations of this section translate to the pointed case and so does the proof. □

5.5 Homotopy Groups

The adjunction gives us the tools to understand homotopy groups of FB in terms of π^*B. This will be a crucial ingredient to proving our main theorem. We introduce the *homotopy groups* of an augmented cdga and define them by $\pi^n(B) = H^n(QB)$.

Theorem 5.11. *Let B be a cofibrant cdga. Then there is a bijective correspondence $\pi_n(FB) \leftrightarrow Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$ for $n \geq 1$. If B is homologically connected then $\pi_0 FB = *$. Furthermore, if $n \geq 2$ the natural bijection above induces an isomorphism of groups $\pi_n(FB) \cong Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$. The isomorphism also holds if $n = 1$ and $B(1) = B(1, 1)$ in the notation of section 3.2.*

While the idea for this proof is simple, we need a number of technical ingredients. First we define $V(n)$ in $\mathscr{D}_0$ to have a single generator in dimension n and trivial multiplication.

Lemma 5.12. *Let $B \in \mathscr{D}_0$. Then homotopy is an equivalence relation on $\mathscr{D}_0(B, V(n))$ and we have a natural bijection $[X, V(n)] \cong Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$.*

Sketch of Proof. For the first assertion we have to show that homotopy is a transitive relation on maps from X to $V(n)$. To do this one first has to show that there is a simplicial structure on $\mathscr{D}_0(X, \nabla(*, *) \otimes V(n))$ obtained just as in our construction of F_0. Furthermore, since $V(n)$ has trivial multiplication, the addition on $V(n)$ induces a group structure on $\mathscr{D}_0(X, \nabla(*, *) \tilde{\otimes} V(n))$ and every simplicial group is a Kan complex, see e.g. [14]. It turns out that the extension properties of a Kan complex precisely give the composite of two homotopies and so homotopy is transitive on $\mathscr{D}_0(B, V(n))$.

The natural map $\phi_B : [B, V(n)] \to Hom_{\mathbb{Q}}(\pi^n B, \pi^n V(n)) = Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$ exists since homotopic maps become identical on applying π^n. We have shown this in the proof of Lemma 4.8. Also, ϕ_B is clearly surjective, so it is left to show that ϕ_B is injective. Observe that ϕ_B is equivalent to $\phi_{B/IB \cdot IB}$, so we can assume that B has trivial multiplication. Now assume $f_* = g_* : \pi^n B \to \pi^n V(n)$. This means there is a chain homotopy $D : X^* \to V(n)^{*-1}$ with $dDx + Ddx = fx - gx$ and with $DX \subset IV(n)$. We will define the map $H : B \to \nabla(1, *) \tilde{\otimes} V(n)$ as follows.

$$H(x) = dt_1 \otimes Dx + 1 \otimes fx - t_1 \otimes fx + t_1 \otimes gx$$

We can check this is a legitimate map and we have $\partial_0 H = g$ and $\partial_1 H = f$. Thus $f \sim g$ and ϕ_b is injective. □

Lemma 5.13. *Given a push-out square in $\mathscr{D}_0$*

$$\begin{tikzcd}
B \arrow[r, "j"] \arrow[d, hook, "i"'] & C \arrow[d, "k"] \\
D \arrow[r, "h"'] & L
\end{tikzcd}$$

where i is a cofibration, we have a long exact sequence

$$\pi^0 B \overset{(i_*,j_*)}{\longrightarrow} \pi^0 C \oplus \pi^0 D \overset{k_*-h_*}{\longrightarrow} \pi^0 L \longrightarrow \pi^1 B \overset{(i_*,j_*)}{\longrightarrow} \cdots$$

Proof. One can show that the square remains a push-out if we apply Q to it [4]. We know a push-out of an injection gives rise to a long exact sequence, so all we have to do is prove that Qi is an injection on QB^n. We will prove this by showing that Qi induces a surjection $Hom_{\mathbb{Q}}(QD^n, \mathbb{Q}) \to Hom_{\mathbb{Q}}(QB^n, \mathbb{Q})$. Introduce the algebra $U(n)$ with generators in degree n and $n-1$, trivial multiplication and zero cohomology. Then we have a natural bijection $\mathscr{D}_0(B, U(n)) \cong Hom_{\mathbb{Q}}(QB^n, \mathbb{Q})$. Since i is a cofibration and the augmentation $\epsilon : U(n) \to \mathbb{Q}$ is a trivial cofibration, we can apply the left lifting property to show that there is a surjection $\mathscr{D}_0(D, U(n)) \to \mathscr{D}_0(B, \mathbb{Q})$. This completes the proof. □

Proof of Theorem. First note that if B is homologically connected it has a minimal model $e_B : M \to B$. FM has a single vertex since M has a unique augmentation, so $\pi_0 FM = *$. We have to show that $\pi_0 FM \cong \pi_0 FB$ but e_B is a weak equivalence, so is a homotopy equivalence by (4.1) and preserved as such by F. Now homotopy equivalences in $\mathscr{S}_0$ are weak equivalences and we are done.

For $n \geq 1$ we can use the adjunction $M : Ho\mathscr{S}_0 \rightleftarrows Ho\mathscr{D}_0 : F$ and the minimal model for the sphere to get the first two bijections of

$$\pi_n FB = [S^n, FB] \cong [B, AS^n] \cong [B, V(n)] \cong Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$$

where $V(n)$, defined as above, is clearly quasi-isomorphic to AS^n. The last bijection is then (5.12).

For the multiplicative structure we first cut down the problem. Assume without loss of generality that B is minimal. By (4.7) we have cofibrations $B(n) \hookrightarrow B$ and $B(n -$

1) $\hookrightarrow B(n)$. We apply Lemma 5.13 to the push-out diagrams for $B \hookleftarrow B(n) \to Q$ and $B(n) \hookleftarrow B(n-1) \to \mathbb{Q}$ and the long exact sequences reduce to isomorphisms $\pi^n X \cong \pi^n B(n)$ and $\pi^n(B(n) \cong \pi^n(B(n)//B(n-1))$, where $B(n)//B(n-1)$ is $B(n)$ modulo the ideal of the inclusion of $B(n-1)$.

So it is enough to prove the result for $B' := B(n)//B(n-1)$. But this is just a tensor product of $\Lambda(x)$. This is clear if $n \geq 2$ and if $n = 1$ this follows from the assumption $B(1) = B(1,1)$, that is, there are no nontrivial differentials on $B(1)$. But $\otimes\Lambda(x)$ has a comultiplication, given by $x \mapsto (x \otimes 1) \oplus (1 \otimes x)$, with counit. This structure is compatible with all the bijections in the previous paragraph, by naturality. It makes all the terms in that chain of bijections into monoids with unit. So we just have to check that it induces the correct multiplication on $[S^n, FB']$ and $Hom_{\mathbb{Q}}(\pi^n B', \mathbb{Q})$. But the comultiplication on B' makes FB' into a monoid and we know (cf. "simplicial set" in [11]) that the multiplication of a simplicial monoid induces the multiplication in every homotopy group. Finally on $Q(B' \otimes B') = (QB' \otimes \mathbb{Q}) \oplus (\mathbb{Q} \otimes QB')$ the comultiplication induces $x \mapsto x \oplus x$ and this gives the additive structure of $Hom_{\mathbb{Q}}(\pi^n B', \mathbb{Q})$. □

6 The Main Theorem

We want to improve on the result from the last chapter and restrict attention to subcategories on which the adjunction gives an equivalence of categories.

6.1 Statement of the Main Theorem

First we will need to make some definitions and quote some results to state precisely what we want to show. We work in the pointed categories $\mathscr{S}_0$ and $\mathscr{D}_0$ of pointed spaces and augmented cdgas respectively.

A space $X \in \operatorname{ob} \mathscr{S}_0$ is *nilpotent* if its fundamental group is nilpotent and all higher homotopy groups are nilpotent as $\pi_1(X)$-modules. A G-module N is nilpotent if the lower central series $N = \Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \dots$ terminates, where we define $\Gamma_{i+1} = \{gn - n \mid g \in G, n \in \Gamma_i\}$. Nilpotency is a generalisation of simply connectedness and simplicity. For a discussion of nilpotent topological spaces see chapter 8^{bis} of [15]. An unpointed connected space is called nilpotent if it is nilpotent for some and thus for any choice of base-point.

We are interested in this condition because it relates to the *Postnikov tower* for a space. Recall that a Postnikov tower is given by the following data:

- Spaces X_n, whose homotopy groups are equal to those of X in dimension $\leq n$ and vanish above dimension n. We call X_n the n-th *section* of the Postnikov tower.
- A sequence of fibrations $p_n : X_n \twoheadrightarrow X_{n-1}$ with fibre $K(\pi_n(X), n)$.
- A sequence of maps $f_n : X \to X_n$ such that $p_n \circ f_n = f_{n-1}$ and f induces isomorphisms on all homotopy groups up to dimension n.

Given the system $X_0 \twoheadleftarrow X_1 \twoheadleftarrow X_2 \twoheadleftarrow \cdots$ we can recover X (up to weak equivalence) as the categorical limit of the sequence, see the entry "Postnikov system" in [11]. We will need the result from [3] that a connected Kan complex X is nilpotent if and only if every $X_n \to *$ factors as a finite sequence of principal fibrations with connected fibres, where X_n is n-th Postnikov section of X. Every principal fibration is a pull-back of the path fibration $\mathscr{P}K(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$, so we can build the Postnikov tower of a nilpotent space from this fibration.

A nilpotent connected space X is *of finite* $\mathbb{Q}$*−type* if the $\mathbb{Q}$-vector spaces $H_n(X, \mathbb{Q})$ are finite-dimensional for each n, or equivalently if $H_1(X, \mathbb{Q})$ and $\pi_n(X) \otimes \mathbb{Q}$ for $n \geq 2$ are finite dimensional. A nilpotent connected space X is rational and of finite $\mathbb{Q}$-type if and only if all of the fibres in the above factorisation of $X_n \to *$ can be chosen to be $K(\mathbb{Q}, n)$, $n \geq 1$. Similarly a homologically connected algebra $B \in \text{ob}\, \mathscr{D}_0$ is *of finite* $\mathbb{Q}$*-type* if $pi^n B$ is finite-dimensional for all n.

Definition. Let $Ho\mathscr{S}_0^{\#}$ denote the full subcategory of $Ho\mathscr{S}_0$ whose objects are connected, nilpotent, rational spaces of finite $\mathbb{Q}$-type. Let $Ho\mathscr{D}_0^{\#}$ denote the full subcategory of $Ho\mathscr{D}_0$ whose objects are cofibrant homologically connected algebras of finite $\mathbb{Q}$-type. Analogously we define $Ho\mathscr{S}^{\#}$ and $Ho\mathscr{D}^{\#}$ as subcategories of $\mathscr{S}$ and $\mathscr{D}$ respectively.

We are now ready to state the main theorem, together with some corollaries, the first of which completes the promised algebraic characterisation of rational homotopy theory.

Theorem 6.1. *The adjunction* $M : Ho\mathscr{S}_0 \rightleftarrows Ho\mathscr{D}_0 : F$ *induces an equivalence of categories between* $Ho\mathscr{S}_0^{\#}$ *and* $Ho\mathscr{D}_0^{\#}$. *The same is true for the adjunction of unpointed categories.*

Corollary 6.2. *There is an equivalence between homotopy classes of rational nilpotent connected CW-complexes of finite* $\mathbb{Q}$*-type and their minimal models.*

Corollary 6.3. *The unit* $\phi_X : X \to FMX$ *of our adjunction is a rationalisation of* X *for every connected nilpotent space* X.

Corollary 6.4. *For a nilpotent connected CW-complex* X *with minimal model* M *we have* $\pi_n(X) \otimes \mathbb{Q} = \pi^n M$ *if* $n \geq 2$.

For the first corollary we recall that $Ho\mathscr{S}$ and Ho**Top** are equivalent and note that $Ho\mathscr{D}_0^{\#}$ is equivalent to its full subcategory of minimal models. The second corollary is a straightforward consequence of the existence of a rationalisation functor which is proved in [3]. The third one is evident from 5.11 and inspection of the proof of the main theorem. Now we need one more ingredient before we can embark on that proof.

6.2 Cohomology of Eilenberg-Mac Lane Spaces

We will have to know the rational cohomology of $K(\mathbb{Q}, n)$ for the main construction. We will proceed by induction and start with

Lemma 6.5. *$K(\mathbb{Q}, 1) = S^1_{\mathbb{Q}}$.*

Proof. Recall the construction of $S^1_{\mathbb{Q}} = \bigvee_k S^1_k \cup_h \amalg_k D^1_k$, with one circle for each natural number k and attaching map $h|_{D_k} : S^1 \to S^1_k \vee S^1_{k+1}$ which is $[S^1_k] - (k+1)[S^1_{k+1}]$ on homology. We can consider a truncated construction $(S^1)^K$, by just considering the first K circles and $K - 1$ disks. Note that $(S^1)^K \cong S^1$ by projecting onto S^1_K. Now observe that any map from S^n into $S^n_{\mathbb{Q}}$ has compact image, so lies in some $(S^1)^K$ and is homotopically trivial if $n \geq 2$.

To show that $\pi_1(S^n_{\mathbb{Q}}) \cong \mathbb{Q}$ note that the fundamental group is commutative by considering $(S^1)^K$ again, and is uniquely divisible and has a single generator over $\mathbb{Q}$. □

For our inductive step we will consider the path fibration $\Omega K(\mathbb{Q}, n) \to \mathscr{P}K(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$, where we note that $\mathscr{P}K(\mathbb{Q}, n) \cong *$ implies that $\Omega K(\mathbb{Q}, n) \cong K(\mathbb{Q}, n - 1)$ by considering the long exact sequences of homotopy groups.

Lemma 6.6. *The cohomology ring of $K(\mathbb{Q}, n)$ over $\mathbb{Q}$ is $\Lambda(x)$ with $\deg x = n$.*

Proof. The result is clear for $K(\mathbb{Q}, 1)$ by the previous lemma. We will use the Leray-Serre spectral sequence for the inductive step. For an introduction to spectral sequences see for example [1] or [15]. Given a principal fibration $F \to E \to B$, with B a connected CW-complex and $\pi_1 B$ acting trivially on the cohomology of F, there is a spectral sequence of algebras $H^p(B, H^q(F)) \Rightarrow H^*(E)$. These conditions are clearly met, so assume the result holds for $K(\mathbb{Q}, 2n - 1)$ and consider the spectral sequence for $K(\mathbb{Q}, 2n - 1) \to * \to K(\mathbb{Q}, 2n)$.

We assume that the only rational cohomology of $K(\mathbb{Q}, 2n - 1)$ is $\mathbb{Q}$ in dimension 0 and $\mathbb{Q}\alpha$ in dimension $2n - 1$. So there are only two nonzero rows, and only the differential d_{2n} can be nontrivial. But the total space has trivial cohomology, so $d_{2n} : E^{0,2n-1}_{2n} \to E^{2n,0}_{2n}$ must be an isomorphism, $E^{2n,0}_{2n} = H^{2n}(K(\mathbb{Q}, 2n), \mathbb{Q})$ is generated by $\beta = d_{2n}\alpha$. But then $E^{2n,2n-1}_{2n} = \mathbb{Q}\alpha\beta$ and there is another isomorphism $d_{2n} : \alpha\beta \mapsto \beta^2$ since $d_{2n}\beta = 0$. Now proceed by induction.

$$\begin{tikzcd}
2n{-}1 & \mathbb{Q}\alpha \arrow[rd, "d_{2n}"] & \mathbb{Q}\alpha\beta \arrow[rd, "d_{2n}"] & \cdots & \\
0 & \mathbb{Q} & \mathbb{Q}\beta & \mathbb{Q}\beta^2 & \cdots \\
 & 0 & 2n & 4n &
\end{tikzcd}$$

Next assume that the rational cohomology of $K(\mathbb{Q}, 2n)$ is the polynomial algebra in one generator of dimension $2n$. By the same argument as above $d_{2n} : E_{2n+1}^{0,2n} \to E_{2n+1}^{2n+1,0}$ is an isomorphism, say $d_{2n+1} : \beta \mapsto \alpha$. Then $d_{2n+1} : E_{2n+1}^{0,4n} \to E_{2n+1}^{2n+1,2n}$ is given by $\beta^2 \mapsto \alpha\beta + \beta\alpha = 2\alpha\beta$, and this is an isomorphism on $\mathbb{Q}$. So $E_{2n+1}^{4n+2,0} = 0$ since otherwise it would live to E^∞, contradicting the triviality of the total space. Again the proof finishes by induction.

$$\begin{tikzcd}
4n & \mathbb{Q}\beta^2 \arrow[rd, "d_{2n+1}"] & \vdots & \vdots \\
2n & \mathbb{Q}\beta \arrow[rd, "d_{2n+1}"] & \mathbb{Q}\alpha\beta \arrow[rd, "0"] & 0 \\
0 & \mathbb{Q} & \mathbb{Q}\alpha & 0 \\
 & 0 & 2n{+}1 & 4n{+}2
\end{tikzcd}$$

□

Remark. While this is the “classical” method for finding the rational cohomology of Eilenberg-Mac Lane spaces, it is worth noting that there is a proof without spectral sequences that uses minimal models and the more general Sullivan models, see § 15 of [7].

6.3 Proof of the Main Theorem

We restrict ourselves to the proof of the pointed version of the theorem, the unpointed version then follows easily. The proof proceeds by “induction up a Postnikov tower”. We want to show the unit of the adjunction is a an isomorphism in the homotopy category, so it is an equivalence in the ordinary categories. The desired property is first established for rational Eilenberg-Mac Lane spaces, then it is shown to be preserved under the pull-backs which are used to construct the Postnikov tower of an arbitrary connected nilpotent rational space. The last step is to show that if the property holds for all Postnikov sections of a space, it holds for the space itself. The dual procedure can be used in the other direction. This construction shows that the Postnikov tower of a space and the decomposition of a minimal model into successive cofibrations are dual, and the resulting objects are in a bijective correspondence up to homotopy.

Example 6.1. Let us look at an example for the duality of Postnikov tower and decomposition of the minimal model. We showed in Example 3.2 that the rational 2-sphere $S^2_{\mathbb{Q}}$ has minimal model $M = \Lambda(x, y \mid dy = x^2)$ where $\deg x = 2, \deg y = 3$. So the decisive step in the construction leads from $\Lambda(x)$ to M. We have the following push-out square (compare diagram 4.1).

$$\begin{tikzcd}
\Lambda(w) \arrow[r, "f"] \arrow[d, "\theta"'] & \Lambda(x) \arrow[d] \\
\Lambda(y, dy) \arrow[r] & M
\end{tikzcd}$$

Where $\deg w = 4$ and we have the maps $f : w \mapsto x^2$ and $\theta : w \mapsto dy$. This gives M as above. Sometimes f is called a *Hirsch extension*. Dually we have the first nontrivial Postnikov section $(S^2_{\mathbb{Q}})_2 = K(\mathbb{Q}, 2)$, whose minimal model is $\Lambda(x)$, and dually to the map $\theta : \Lambda(w) \to \Lambda(y, dy)$ we have the path fibration $\pi : \mathscr{P}K(\mathbb{Q}, 4) \to K(\mathbb{Q}, 4)$. We get the following pull-back, where k is the second k-invariant of the Postnikov tower for $S^2_{\mathbb{Q}}$.

$$\begin{tikzcd}
S^2_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathscr{P}K(\mathbb{Q}, 4) \arrow[d, "\pi"] \\
(S^2_{\mathbb{Q}})_2 \arrow[r, "k"'] & K(\mathbb{Q}, 4)
\end{tikzcd}$$

Let $\phi_X : X \to FMX$ and $\psi_B : B \to MFB$ denote the units of the adjunction $M : Ho\mathscr{S}^{\#}_0 \rightleftarrows Ho\mathscr{D}^{\#}_0 : F$. We can obtain them from the units θ_X, η_B of $A : Ho\mathscr{S} \leftrightarrows Ho\mathscr{D} : F$. Observe that ϕ_X is the composite

$$X \xrightarrow{\theta_X} FAX \xrightarrow{Fe_X} FMX$$

and ψ_B is any lift in the following diagram. The lift exists since B is cofibrant.

$$\begin{tikzcd}
 & MFB \arrow[d, "e_{FB}"] \\
B \arrow[ur, dashed, "\psi_B"] \arrow[r, "\eta_B"'] & AFB
\end{tikzcd}$$

Lemma 6.7. *If $X = K(\mathbb{Q}, n)$ then $MX \in ob\ Ho\mathscr{D}_0^{\#}$ and $\phi_X : X \to FMX$ is an isomorphism in $Ho\mathscr{S}_0^{\#}$*

Proof. It follows from (6.6) that AX has a minimal model $\Lambda(x)$ where $\deg x = n$, the weak equivalence is given by mapping x to the generator of the single generator of the de Rham cohomology ring. This proves the first part of the lemma. Next it follows from (5.11) that for $i \geq 1$

$$\pi_i(FMX) = Hom_{\mathbb{Q}}(MX, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

So we know that FMX is a $K(\mathbb{Q}, n)$. We still have to show that the map ϕ_X is an isomorphism. But we know that $M\phi_X$ has a right inverse, and this shows ϕ_X is surjective on cohomology by the following diagram, where the vertical arrows are isomorphisms.

$$\begin{tikzcd}
HMFMX \arrow[r, "HM\phi_X"] \arrow[d, "\rho\circ e_{FMX}"'] & HMX \arrow[d, "\rho\circ e_X"] \\
HFMX \arrow[r, "H\phi_X"'] & HX
\end{tikzcd}$$

But the cohomology rings of X and FMX are generated by a single generator in degree n, so the map on cohomology is determined by what happens in degree N. Now any surjection $\mathbb{Q} \to \mathbb{Q}$ is an isomorphism, so we have an endomorphism of X inducing an isomorphism on cohomology. Since X is nilpotent, this is an automorphism by the generalised Whitehead theorem (Corollary 8^{bis}.24 in [15]). □

The proof of the dual version of (6.7) follows similar ideas and can be found in §10 of [2].

Lemma 6.8. *If $B = \Lambda(x)$ then $FB \in ob\ Ho\mathscr{S}_0^{\#}$ and $\psi_B : B \to MFB$ is an isomorphism in $Ho\mathscr{D}_0^{\#}$.*

For the following lemma we need to take a couple of results for granted. We will give some indication as to how they are proved. More details can be found in [2], the necessary background on spectral sequences is available in [15]. The results will be concerned with the following situation. Consider a commutative square in $\mathscr{D}_0$.

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & C \arrow[d] \\
D \arrow[r] & L
\end{tikzcd}$$

What can we say about the cohomology of L, when we know B, C and D? We can construct a map from the differential Tor functor $\mathrm{Tor}_B(C, D)$ to HL, see section 7.2 in [15]. If this map is an isomorphism the square is called an *Eilenberg-Moore square*.

Lemma 6.9. *Let E be the pullback of $Y \to X \leftarrow Z$ in $\mathscr{S}_0$, where X is simply connected and all objects are connected of finite $\mathbb{Q}$-type. Then by applying A to the fibre square we obtain an Eilenberg-Moore square.*

Idea of proof. The statement is true if we apply the singular cochain functor C^* instead of A. This is the Eilenberg Moore spectral sequence, which is the spectral sequence of the (filtered) bar resolution $B(C^*Y, C^*X, C^*Z)$. We already met the natural transformation $\rho : A \to C^*$, so we have the following situation.

$$\begin{tikzcd}
& AX \arrow[rr] \arrow[dl, "\rho_{AX}"] \arrow[dd] & & AY \arrow[dl, "\rho_{AY}"] \arrow[dd] \\
C^*X \arrow[rr] \arrow[dd] & & C^*Y \arrow[dd] & \\
& AZ \arrow[rr] \arrow[dl, "\rho_{AZ}"] & & AE \arrow[dl, "\rho_{AE}"] \\
C^*Z \arrow[rr] & & C^*E &
\end{tikzcd}$$

So it is enough to show that ρ induces an isomorphism $\mathrm{Tor}_{AX}(AY, AZ) \cong \mathrm{Tor}_{C^*X}(C^*Y, C^*Z)$. However, this is not at all obvious since ρ is not multiplicative and thus not a map of algebras. The idea now is that it is enough for ρ to be multiplicative "up to homotopy". We can extend the map ρ_{AX} to a map $B(AX) \to B(C^*X)$, this is done in chapter 3 of [2]. We call ρ_{AX} a strongly homotopy multiplicative map. Since ρ_{AY} and ρ_{AZ} are also strongly homotopy multiplicative we get a map $B(AY, AX, AZ) \to B(C^*Y, C^*X, C^*Z)$, see section 3 of [10]. Because the filtration of the bar resolution is nice (that is, exhaustive and strongly convergent), the isomorphism that ρ induces on the E_2-term $\mathrm{Tor}_{HAX}(HAY, HAZ)$ leads to an isomorphism of the extended map on $HB(AY, AX, AZ) = \mathrm{Tor}_{AX}(AY, AZ)$ and this is what we needed. □

The next two results also follow from spectral sequence arguments, however they need less input. In the first case one uses the spectral sequence of the bar resolution filtered by internal degree. The second case follows from a standard result about the Eilenberg-Moore spectral sequence (Theorem 7.7 in [15]).

Lemma 6.10. *Consider the push-out square of $C \leftarrow B \to D$ in $\mathscr{D}_0$. Suppose B is simply connected and $(C, 0)$ is free as a module over $(B, 0)$, where $(C, 0)$ is the graded module obtained from C by forgetting the differential. Then the square is an Eilenberg-Moore square.*

Lemma 6.11. *A map of Eilenberg-Moore squares in $\mathscr{D}_0$ as below inducing isomorphisms $HB \cong HB', HC \cong HC', HD \cong HD'$ also induces an isomorphism $HL \cong HL'$*

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & C \arrow[d] \arrow[r, phantom, "\Rightarrow"] & B' \arrow[r] \arrow[d] & C' \arrow[d] \\
D \arrow[r] & L & D' \arrow[r] & L'
\end{tikzcd}$$

Now we are ready to prove the inductive step in the proof of our main theorem.

Lemma 6.12. *Consider the following fibre square in $Ho\mathscr{S}^{\#}_0$.*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathscr{P} \arrow[d] \\
Y \arrow[r, "k"'] & K(\mathbb{Q}, n)
\end{tikzcd}$$

Where π is the usual path fibration. If $MY \in$ ob $Ho\mathscr{D}^{\#}_0$ and $\phi_Y : Y \cong FMY$ then $MX \in$ ob $Ho\mathscr{D}^{\#}_0$ and $\phi_X : X \cong FMX$.

Proof. We want to apply A to the given square and then use (6.9) and properties of $AK(\mathbb{Q}, n), A\mathscr{P}$ and AY to deduce properties of AX. But the image under A is not in general cofibrant, so we need to find cofibrant replacements for all objects.

$$\begin{tikzcd}
\Lambda(x) \arrow[rrr] \arrow[ddd, "\theta"'] \arrow[dr, "m"] & & & S \arrow[dl, "s"'] \arrow[ddd] \\
& AK(\mathbb{Q}, n) \arrow[r, "Ak"] \arrow[d] & AY \arrow[d] & \\
& A\mathscr{P} \arrow[r] & AX & \\
\Lambda(y, dy) \arrow[rrr] \arrow[ur, "n"] & & & R \arrow[ul, "r"']
\end{tikzcd}$$

In this diagram $\theta : x \mapsto dy$ as before and all the diagonal arrows are weak equivalences. We construct it as follows. Since we know the minimal model for $K(\mathbb{Q}, n)$ and $\mathscr{P}$ is contractible we get the left square. We then factor $Ak \circ m : \Lambda(x) \to AY$ as a cofibration followed by a weak equivalence. Thus we get S, a cofibrant replacement for AY. Finally let R be the push-out. Since θ is a cofibration, so is its push-out and R is cofibrant. Then r exists by the universal property of a push-out. Now both the outer square (by Lemma 6.10) and the inner square (by Lemma 6.9) are Eilenberg-Moore squares. So it follows from (6.11) that r is a weak equivalence. Now we have $R \cong MX$. So to show $MX \in \text{ob}\, Ho\mathscr{D}_0^{\#}$ it is enough to show that $R \in \text{ob}\, Ho\mathscr{D}_0^{\#}$. But this follows, using that $S \cong MY \in \text{ob}\, Ho\mathscr{D}_0^{\#}$, from (5.13).

Next we form the adjoint of our diagram.

$$\begin{tikzcd}
F\Lambda(x) & & & FS \arrow[lll] \\
& K(\mathbb{Q},n) \arrow[ul, "m^\#"] & Y \arrow[l] \arrow[ur, "s^\#"'] & \\
& \mathscr{P} \arrow[u] \arrow[dl, "n^\#"'] & X \arrow[l] \arrow[u] \arrow[dr, "r^\#"] & \\
F\Lambda(y,dy) \arrow[uuu] & & & FR \arrow[lll] \arrow[uuu]
\end{tikzcd}$$

Now we note that since X is cofibrant $\phi_X : X \to FMX$ is a weak equivalence in $\mathscr{S}_0$ if and only if $r^\# : X \to FR$ since the former is the composition of the latter with $FR \cong FMX$ which is a homotopy equivalence by (5.5) . The same is true for ϕ_Y and $r^\#$, $\phi_{\mathscr{P}}$ and $n^\#$ and $\phi_{K(\mathbb{Q},n)}$ and $m^\#$. Now ϕ_Y is a weak equivalence by assumption, $\phi_{\mathscr{P}}$ is trivially a weak equivalence and $\phi_{K(\mathbb{Q},n)}$ is an equivalence by (6.7). So $s^\#$ is an equivalence by (6.11), and this implies that $\phi_X : X \cong FMX$ in $Ho\mathscr{S}_0^\#$. □

Again, we will only state the dual version of this lemma. The proof follows the same ideas, but is easier since we do not have to worry about cofibrant replacement.

Lemma 6.13. *Consider the following push-out square in* $Ho\mathscr{D}_0^\#$.

$$\begin{tikzcd}
\Lambda(w) \arrow[r] \arrow[d, hook, "\theta"'] & B \arrow[d] \\
\Lambda(y,dy) \arrow[r] & N
\end{tikzcd}$$

Where θ *is the cofibration defined earlier. If* $FB \in ob\ Ho\mathscr{S}_0^\#$ *and* $\psi_B : B \cong MFB$ *then* $FN \in ob\ Ho\mathscr{S}_0^\#$ *and* $\psi_N : X \cong MFN$.

Proof of Theorem 6.1. Let $X \in \mathscr{S}^\#$. We first have to show $MX \in \text{ob}\, Ho\mathscr{D}_0^\#$ and $\phi_X : X \cong FMX$. Consider X_n, the n-th Postnikov section of X. Its homotopy type can be obtained from $K(\pi_1(X), 1)$ in finitely many steps as in (6.12). So by (6.12) and (6.7) the assertion holds for X_n. Also we have $H^i(X) \cong H^i(X_n)$ for $i \leq n$ via the induced map. By the Whitehead theorem for minimal models (4.10) this implies $\pi^i MX \cong \pi^i MX_n$ for $i \leq n$ and thus $MX \in \text{ob}\, \mathscr{D}^\#$.

Now consider the tower of fibrations $X_0 \twoheadleftarrow X_1 \twoheadleftarrow X_2 \twoheadleftarrow \cdots$. We have $X \cong \varprojlim X_n$. We apply A to get a tower of cofibrations and replace them by their minimal models MX_i. We define $MX = \varinjlim MX_n$ and have the diagram below on the left. We see that the map $MX \to AX$ is a quasi-isomorphism by noting that $H^i MX_n \cong H^i M$ and $H^i AX_n \cong H^i AX$ for $i \leq n$. So MX is indeed a minimal model for X. Now we carry our diagram across the adjunction to get the diagram on the right.

$$\begin{tikzcd}
AX & & \\
& AX_n \arrow[ul] \arrow[r, hook] & AX_{n+1} \arrow[ull] \\
& MX_n \arrow[u, "\sim"] \arrow[r, hook] \arrow[dl] & MX_{n+1} \arrow[u, "\sim"'] \arrow[dll] \\
MX \arrow[uuu, "\sim"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr] \arrow[drr] \arrow[ddd, "\phi_X"'] & & \\
& X_n \arrow[d, "\phi_{X_n}"'] & X_{n+1} \arrow[l, two heads] \arrow[d, "\phi_{X_{n+1}}"] \\
& FMX_n & FMX_{n+1} \arrow[l, two heads] \\
FMX \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Now $FMX_0 \twoheadleftarrow FMX_1 \twoheadleftarrow \cdots$ is a tower of fibrations whose fibres are Eilenberg-Mac Lane spaces. So it is fibred Postnikov system and a Postnikov tower for its limit. Since F takes colimits to limits we have $\varprojlim FMX_n = F \varinjlim MX_n = FMX$. So by Lemma 6.12 we have isomorphisms

$$\pi_i X \cong \pi_i X_n \cong \pi_i FMX_n \cong \pi_i FMX.$$

And this proves that $\phi_X : X \to FMX$ is an isomorphism in $Ho\mathscr{S}_0^\#$.

Dually we have to show that for $B \in Ho\mathscr{D}_0^\#$ we have $FB \in \text{ob}\, Ho\mathscr{S}_0^\#$ and $\psi_B : B \cong MFB$. This time we consider B as a colimit of $B_0 \hookrightarrow B_1 \hookrightarrow B_2 \hookrightarrow \cdots$. We apply F and find that the FB_n form a fibred Postnikov system with $FB = \varprojlim FB_n$. Carry the identity on this sequence across the adjunction to obtain the following diagram.

$$\begin{tikzcd}
B \arrow[dr] \arrow[drr] \arrow[ddd, "\psi_B"'] & & \\
& B_n \arrow[r, hook] \arrow[d, "\psi_{X_n}"'] & B_{n+1} \arrow[d, "\psi_{X_{n+1}}"] \\
& MFB_n \arrow[r, hook] & MFB_{n+1} \\
MFB \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Now we know we have isomorphism $\pi_i FB_n \cong \pi_i FB$ for $i \leq n$. This shows that $FB \in$ ob $Ho\mathscr{S}_0^\#$. It also implies, by the Whitehead theorem and the universal coefficient theorem, that $H^i FB \cong H^i FB_n$ and thus $H^i MFB \cong H^i MFB_n$. Together with $H^i B \cong H^i B_n$ and Lemma 6.13 this shows that ψ_B is an isomorphism. So we have shown that our adjunction restricts to $M : Ho\mathscr{S}_0^\# \leftrightarrows Ho\mathscr{D}_0^\# : F$ and since both units of this adjunction are isomorphisms we have an equivalence of categories. $\square$

7 Formality

Now that we have proved how useful minimal models are, it would be good to know how much effort we have to put in to obtain them. So far the only way we have is to (more or less) calculate the de Rham complex. It would be very convenient if just knowing the cohomology would give us enough information to obtain a minimal model. While we know this cannot hold true in general, there is a class of examples where the cohomology determines the rational homotopy type.

Definition. A cdga B is *formal* if we have a quasi-isomorphism $B \cong HB$, where we turn HB into a cdga by defining the differential to be 0 everywhere. We will say that a topological space X is *formal* if its minimal model is formal.

Example 7.1. Most of the examples we looked at so far were formal. Looking back at the calculations we find that S^n, $K(\mathbb{Q}, n)$, $(S^3 \times S^2) \# (S^3 \times S^2)$ and $(S^1 \times S^2) \# (S^1 \times S^2)$ are all formal. However, it is easily seen that the space N_Γ constructed in chapter 3 is not formal.

7.1 Kähler Manifolds are Formal

We will prove that every Kähler manifold is formal, that is, its rational homotopy type is determined by its de Rham cohomology ring. This was first shown by Deligne, Griffiths, Morgan and Sullivan in [5] and we are going to follow their proof. The result is a surprisingly straightforward consequence of the dd^c-lemma, which is a result of the Hodge theory of Kähler manifolds. Recall that on any complex manifold we have a real differential operator $d^c = i(\overline{\partial} - \partial) = J^{-1}dJ$ satisfying $d^c d^c = 0$ and $dd^c = -d^c d$.

Lemma 7.1. *Let M be a Kähler manifold and η a differential form on M satisfying $d^c\eta = 0$ and $\eta = d\gamma$. Then $\eta = dd^c\beta$ for some β.*

For a proof see [13]. There is a dual $d^c d$-lemma obtained by applying the dd^c lemma to $J\eta$. Our result is an immediate consequence of this lemma and our final theorem.

Theorem 7.2. *Every compact complex manifold M satisfying the dd^c-lemma is formal.*

Proof. We will show that the de Rham complex of M is quasi-isomorphic to the cohomology of M. We look at the following complexes:

a) the de Rham complex $(\Omega M, d)$,

b) the subcomplex of d^c-closed forms $(\Psi M, d)$,

c) the subquotient of d^c-closed forms modulo d^c-exact form $(H_{d^c}M, d)$.

We have the obvious inclusion and quotient maps $(\Omega M, d) \xleftarrow{i} (\Psi M, d) \xrightarrow{q} (H_{d^c}M, d)$. It is now a matter of diagram chasing and using the dd^c-lemma to show that i and q are quasi-isomorphisms and that d is 0 on $H_{d^c}M$.

a) d is 0 on H_{d^c}. We have to show that given a d^c-closed form $\alpha \in \Psi M$ we have $d\alpha = d^c\gamma$ for some $\beta \in \Psi M$. But $d\alpha$ satisfies the conditions of the dd^c-lemma, so we have $d\alpha = dd^c\beta = d^c(-d\beta)$ and $[d\alpha] = 0$ in $H_{d^c}M$.

b) i is injective. Let η be a form in ΩM that is d^c-closed and satisfies $\eta = d\gamma$. Then the dd^c-lemma applies and we have $\eta = d^c(-d\beta)$.

c) i is surjective. Let α represent a homology class in $H\Omega M$. We need a d^c-closed form that is equal to α on cohomology. Note that $d^c\alpha$ satisfies the conditions of the dd^c-lemma, so we have $d^c\alpha = dd^c\beta$ and $\alpha + d\beta$ is d^c-closed.

d) q is surjective. Let $[\alpha]$ be an element of $H_{d^c}M$. Then the representative α is d^c-closed and we need a d^c-homologous element that is d-closed. But $d\alpha = dd^c\beta$ by another application of the dd^c-lemma and so $\alpha - d^c\beta$ will do the job.

e) q is injective. Let α be d-closed and $\alpha = d^c\gamma$. We have to show α is d-exact. This time we use the d^cd-lemma to obtain $\alpha = d^cd\beta$ and the result readily follows. □

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