# Report on the 2023 Balkan Mathematical Olympiad

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With input from Dominic Yeo

The Balkan Mathematical Olympiad is a competition run by the 11 member countries of the Mathematical Association of South-Eastern Europe. This year was the $40^{\text{th}}$ edition, and was held in Antalya, Turkey, from the $8^{\text{th}}$ to the $13^{\text{th}}$ of May. The United Kingdom has participated since 2005 and was one of 10 guest countries participating this year. The UK's approach to this competition is somewhat different to that of the Balkan countries. We have a self-imposed rule that no student may participate in this competition more than once, or if they have already attended the IMO. This allows us to give more students an experience of an international competition, as well as training younger students for competitions in future years. Thus only two of this year's team are members of the squad of 10 from which the IMO team will be picked. The team was picked based on a pair of $4\frac{1}{2}$ hour exams sat at Trinity College, Cambridge, during our training camp in March and April.

The team members were:

| | |
|---|---|
| Anthony Goncharov | Colyton Grammar School |
| Sam Griffiths | Kingswinford Academy |
| Elsa Lin | Westminster School |
| Neil Prabhu | St Paul's School |
| Mikaeel Shah | Isleworth and Syon School |
| Julia Volovich | Hills Road Sixth Form College |

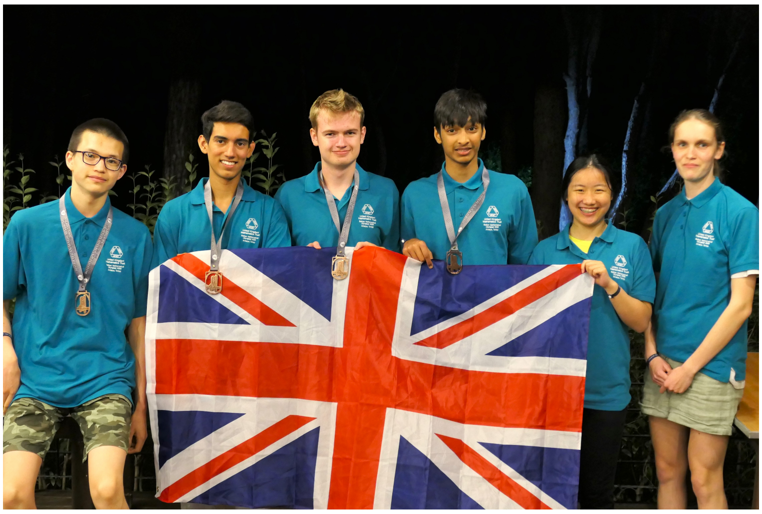
Anthony is in year 13, Sam in year 10, and the rest in year 12. The academic staff were myself, as Leader, and Dominic Yeo, as Observer A. For our purposes, there is little distinction between these roles, other than who does the paperwork. Ina Hughes was Observer C, responsible for looking after the students while on the trip. This was my second year attending as an adult, having been a contestant myself in Albania in 2016, where Dominic was leader.

The Olympiad itself consists of a single $4\frac{1}{2}$ hour paper with one question form each of the standard areas of olympiad maths: Algebra, Combinatorics, Geometry and Number Theory. These should be in order from easy to hard (where easy means hard, and hard means incredibly difficult). However, opinions on difficulty vary significantly. Many students from the Balkans have been training for olympiad maths for years and do lots of geometry at school, so know many useful theorems and lemmas. Our students do not have this experience (Sam had first been to an olympiad event only 6 weeks previously) and so rely on making up the difference in questions with less knowledge required.

The results of the UK team were:

| | P1 | P2 | P3 | P4 | Total | |
|---|---|---|---|---|---|---|
| Anthony Goncharov | 10 | 7 | 3 | 1 | 21 | Bronze |
| Sam Griffiths | 10 | 0 | 2 | 8 | 20 | Bronze |
| Elsa Lin | 10 | 1 | 2 | 1 | 14 | Honourable Mention |
| Neil Prabhu | 10 | 10 | 2 | 0 | 22 | Bronze |
| Mikaeel Shah | 10 | 7 | 0 | 1 | 18 | Bronze |
| Julia Volovich | 8 | 0 | 1 | 1 | 10 | |

This placed the UK $15^{\text{th}}$ out of 22 participating teams, with a score of 105/240. The top three teams were Romania (198), Turkey (184) and Bulgaria (178). The medal cut-offs were 32 points for a Gold, 31 for a Silver, and 17 for a Bronze. Contestants who did not receive a medal but got full marks on a question got an Honourable Mention.



This report consists of two sections: a discussion of the problems, and a diary of events from the leader's perspective, in something close to chronological order.

## The paper

### Problem 1 (Proposed by North Macedonia)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$xf(x + f(y)) = (y - x)f(f(x)).$$

### Problem 2 (Proposed by the United Kingdom)

In triangle $ABC$, the incircle touches sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. Assume there exists a point $X$ on the line $EF$ such that

$$\angle XBC = \angle XCB = 45^\circ.$$

Let $M$ be the midpoint of the arc $BC$ on the circumcircle of $ABC$ not containing $A$. Prove that the line $MD$ passes through $E$ or $F$.

### Problem 3 (Proposed by Greece)

For each positive integer $n$, denote by $\omega(n)$ the number of distinct prime divisors of $n$ (for example, $\omega(1) = 0$ and $\omega(12) = 2$). Find all polynomials $P(x)$ with integer coefficients, such that whenever $n$ is a positive integer satisfying $\omega(n) > 2023^{2023}$, then $P(n)$ is also a positive integer with

$$\omega(n) \geq \omega(P(n)).$$

### Problem 4 (Proposed by Romania)

Find the greatest integer $k \leq 2023$ for which the following holds: whenever Alice colours exactly $k$ numbers of the set $\{1, 2, \ldots, 2023\}$ in red, Bob can colour some of the remaining uncoloured numbers in blue, such that the sum of the red numbers is the same as the sum of the blue numbers.

## Commentary on the problems

What follows here is not meant to be a set of official solutions. The aim is to discuss what we thought of the problems, and how you might go about solving them in practice. We aim to highlight what we consider to be the key steps and motivation. They are therefore missing the details necessary in a solution, and not written as you would when trying to present it in a competition. Any students reading are advised to skip this section unless they have had a serious attempt at solving the problems.

## Problem 1

This was one of our favourite easy problems, and our feeling was that it was slightly harder than P1 can sometimes be. Our students had all had training on functional equations at Trinity Camp in March, and so we were pleased to see a near-perfect set of solutions.

There were a number of approaches to this problem. There are two distinct families of solutions, $f(x) = 0$ and $f(x) = c - x$ for some $c$. Thus we expect a solution to split in to cases at some point. The neatest involved splitting based on whether $f(f(x))$ is ever non-zero, and this is what Anthony and Neil did. The other students used similar ideas in their solutions.

To begin with, we try playing around with some substitutions. Each side is a product of terms, so it would be nice to force each side to be zero. We can do this, by letting $x = 0$ and deducing that $f(f(0)) = 0$, and by letting $y = x$, and deducing that $xf(x + f(x)) = 0$ for all $x$. This latter immediately tells us that $f(x + f(x)) = 0$ for non-zero $x$. It is possible from here to work with non-zero $x$, and work out what to do with $x = 0$ later, but it will help to note that $f(x + f(x)) = 0$ is also true for $x = 0$.

Now that we have an equation with the left hand side just consisting of one term, inside an $f$, and the right being simple, it feels injectivity would be helpful (indeed, we can write $f(x + f(x)) = f(f(0))$, which begs you to apply injectivity). If for now we assume $f$ is injective, we find $x + f(x) = c$ for some constant $c$, and hence $f(x) = c - x$, which works for any $c$.

Now let's try to prove injectivity. Suppose $f(y) = f(y')$. Then the left hand side of the given equation is the same for $(x, y)$ and $(x, y')$, so the right hand sides must also be equal. This gives injectivity, as long as $f(f(x))$ can be non-zero. So to finish, we need to deal with the case $f(f(x)) = 0$ for all $x$, and would like to deduce $f(x) = 0$. As $f(f(x))$ appears in the equation, we have reason to be hopeful that this will work.

Assuming $f(f(x)) = 0$ for all $x$, we deduce that $xf(x+f(y)) = 0$. Now when $x \neq 0$, we have $f(x+f(y)) = 0$. But this is also true when $x = 0$ by our assumption. Now $x + f(y)$ can take any value, so we have $f(x) = 0$ for all $x$.

## Problem 2: Discussion by Dominic Yeo

This problem was written by our friend and colleague Sam Bealing, and means that the UK has now contributed the geometry problem at four successive Balkan olympiads!

The key to this problem is identifying what aspect of the configuration leads to *two* possible outcomes. The problem is symmetric in the roles of $\{B, C\}$, so at some point that symmetry must be broken. It turns out that the two cases arise from a fact, which is certainly not 'well-known' in the UK, but which some students from the Balkan countries are taught as *The Iran Lemma.*

The point $X$ lies on line $EF$ and satisfies $\angle BXC = 90°$. There are certainly at most two such points, since the angle condition is described by a circle (with diameter $BC$). The Iran Lemma says that there are two such points, given by the intersections of the angle bisectors $BI$ and $CI$ with $EF$.

To prove the Iran lemma, one can extend $BI$ to point $X'$ on $EF$. The goal is to prove $\angle BX'C = 90°$, which is equivalent to showing that $IEXC$ is cyclic (and point $D$ also lies on this circle). Directly calculating that $\angle FXC = \angle C/2$ is one way to verify this.

In any case, with the Iran Lemma, for the configuration of the original problem, we must have $X$ on either $BI$ or $CI$, and thus either $\angle B$ or $\angle C$ is $90°$, which explains the two cases. If $\angle B = 90°$, then $MD$ passes through $F$, which can be completed in a few ways, including exploiting the fact that $BFID$ is a square!

Fortunately, there were a few other ways to approach the problem that didn't involve knowing or directly deriving the Iran Lemma. Neil showed that $XEMC$ is cyclic (and equivalently for $XBMF$), leading to several angles of measure 45°. Then, by studying $EIMC$, one shows that it is either cyclic, or a kite with axis of symmetry $EM$. It turns out that one of $EIMC$ and $FBMI$ is cyclic (also containing $X$), and the other is a kite, from which it is a short angle chase to finish directly.

## Problem 3

This we felt was a medium-hard problem, but opinions in the jury room varied. The solutions depend on knowledge of some non-trivial results, such as Dirichlet's theorem, so it was a test of training as well as problem-solving. Our students did not all know this result, and certainly were not used to applying it. The more experienced students on our IMO squad, and some of the other teams, would be more familiar.

To begin with, we note that certain constant polynomials and $P(x) = x^k$ do work. At this point my instinct was that no more would work. My feeling was that I don't really know how to put an upper bound on the number of factors $P(n)$ has unless it factorises, and even then the factorisation of, say, $n + 1$ is not obviously boundable in terms of that of $n$.

To prove other polynomials don't work, I want an $n$ whose prime factors I know exactly, and $P(n)$ to have as many factors as possible. I also feel that $2023^{2023}$ is likely to just be an arbitrary large number, so for now let's just have $n$ a product of $k$ distinct primes, which we will pick later. We want $P(n)$ to be divisible by lots of primes $q$. This is equivalent to $P(n) \equiv 0 \mod q$, which will be equivalent to some condition on $n \mod q$.

This feels like a good place to be. We want $n$ to satisfy lots of congruences modulo different primes – this is doable by the Chinese Remainder Theorem. Specifically, we want to pick some primes $p_i$, satisfying congruence conditions modulo each of the primes $q$, which we can do by Dirichlet's theorem. $n$ is then the product of the $p_i$. We're also going to need $P(x)$ to have a root modulo $q$ for lots of primes $q$. This is in fact Schur's Theorem:

**Theorem.** *Let $Q(x)$ be a non-constant polynomial with integer coefficients. Then there are infinitely many primes $q_i$, and integers $n_i$, such that $q_i$ divides $Q(n_i)$.*

This was assumed to be well-known in the official solutions, but wasn't to our students. It's a nice exercise to prove it if you haven't seen it before.

So we have a sketch of a solution, with some caveats. It won't work if the polynomial is constant, as we can't apply Schur's Theorem. Dirichlet may also fail, if we don't get the right coprimality (this is what happens in the $P(x) = x^n$ case). We also haven't considered the positivity condition at all – perhaps this will be needed.

Let's try to work out the details. Given we know about certain issues with Dirichlet in the perfect power case, which stem from 0 being a root modulo $q$, let's write $P(x) = x^m Q(x)$ for $Q(0) \neq 0$. We aim to show that $Q(x)$ is constant, and then it is easy to show that the constant is 1 or $m$ is 0. It is clear that if $P$ satisfies the condition then so does $Q$. Now assume $Q(x)$ is non-constant, and using Schur, take a large set of primes which do not divide $Q(0)$, but that each divide $Q(n_i)$ for some $n_i$. I will leave it as an exercise to construct another set of primes whose product is $n$, with $Q(n)$ divisible by each of the previous primes, and thus finish the problem.

It is also possible to do the $P(0) = 0$ case separately, which some of our students did and got marks for. Here it suffices to take $n$ to be a large product of primes coprime to $Q(0)$, and argue $Q(n)$ must have a factor that doesn't divide $n$. The rest can be finished with a slightly simplified version of the above, or by an induction argument, replacing $2023^{2023}$ with any positive integer.

## Problem 4: Discussion by Dominic Yeo

This problem was felt to be hard by the leaders during the problem selection day, and was found hard by the contestants too. A first step is to consider what happens if Alice chooses the largest $k$ numbers. Clearly if this turns out to give *more than half* the total sum of all numbers then Bob will be unable to match Alice using what remains. This shows that $k \geq 593$ does *not* work.

What is surprising is that this turns out to be the only constraint. With $k = 592$, for all Alice's choices, Bob can manage to match them. (An alternative version that would be less tractable for Bob is to constrain Alice's total sum.) This makes it an interesting problem even if the known proofs all require some technical steps.

In the official solution to the problem, the main method is to consider pairs

$$(1, 2023),\ (2, 2022),\ (3, 2021),\ \ldots,\ (1011, 1013),$$

and similarly for pairs summing to $2024 + b$ for some $b \leq 2023$. Alice can only have blocked up to 592 of these pairs, which means Bob can form any number between 2024 and about $400 \times 2024$. In particular, Bob doesn't need to make totals $\leq 2024$ since Alice chooses 592 numbers which must have sum much larger than 2024.

At this point, a neat observation is that if Alice's sum is $\alpha$, and $S$ is the total sum $\frac{2023\times 2024}{2}$, then Bob can choose a disjoint set with sum $\alpha$ if and only if he can choose a disjoint set with sum $S - 2\alpha$. So the method of the previous paragraph covers $2024 \leq S - 2\alpha \leq\sim 400 \times 2024$ as well.

There remains the outstanding case when Alice's sum $\alpha$ is close to $S/2$, and this can be handled by some more specific pairings.

The Problem Selection Committee suggested (not unreasonably) that they believed this problem could not be addressed using a greedy algorithm, where Bob mainly proceeds by choosing the largest possible remaining value to colour. However, our UK contestant Sam managed to come up with a solution along these lines. The key detail is the non-standard choice of when Bob should stop choosing greedily. As it differs substantially from the standard solution, his proof, as neatened and corrected by Dominic, is given in the appendix.

# Leader's diary

What follows is a light-hearted diary of my experiences during the week. In summary, it was a very enjoyable week for all, and well-organised.

## Day 0: Sunday 7$^{\text{th}}$ May

Due to the bank holiday, it was suggested that we stay at Stansted the night before our flight to Turkey, hence this report begins on day 0. For me, the day began in Braithwaite, Cumbria, where I had been enjoying the coronation weekend far away from London. After a quick run in the hills, I caught an afternoon train from Penrith to Stansted via London. On arrival at the Holiday Inn Express, I met Ina and the students, who had just gathered in the hotel restaurant. I noticed that the team seemed excited for the competition but not too stressed about the test, and that they seemed to be getting on well. All of these are good signs for an enjoyable competition. We ordered some rather large pizzas, the first of many filling meals during the week, and I distributed matching blue polo shirts to all. Dominic then joined us, and distributed several flags purchased in a coronation sale – a 'Union Jack' and two smaller 'Large Union Jacks' – and a book of past questions and solutions. After finishing the food, conversation turned to maths. The students were keen to discuss the practice paper we had set, as well as a hard question from the Team Selection Test. The adults retired at about 9:30 as it had been a long day for all of us, leaving the students doing maths in the restaurant with instructions to go to bed soon.

## Day 1: Monday 8$^{\text{th}}$ May

I came down to breakfast to find the students sat around the same table I had left them at, again doing maths. I have to believe them when they tell me they had slept in between. We took an 8:30 shuttle bus to the airport and proceeded to check-in, where different members of staff gave contradictory information about whether to check in individually or together. After a slight issue with Elsa's visa, I was able to bring up proof of our return flight tickets and details of the hotel. This satisfied the check-in agent and we were all able to get on the flight. In the security queue, the weakness of my attempts to learn Turkish on Duolingo were exposed, as it transpired I couldn't remember how to say thank you. Fortunately

we were overheard by a man who could tell us – teşekkür ederim to him for this. Anthony taught the others to play contact, which as last year quickly turned to naming chemical elements. On the flight the students worked on various problems, though geometry is made harder by the ban on compasses on planes. Geoff's report from 2012, when it was also held in Antalya, had led us to expect rowdy golfers on the flight, and we were relieved not to find any.

After 5 hours on the plane and a slow baggage reclaim we made it to arrivals to be greeted by a member of Olympiad staff about an hour later than she had expected. As Turkey is outside the EU, data roaming charges are still alive and well - for one student it would have been £15 per MB! We all decided to switch data off and rely on finding wifi. The time away from the hotels was therefore without internet, which was a strange experience for many. We were then driven in a minibus to our hotels, with Dominic and myself dropped off at the Sherwood Exclusive, and Ina and the team at the Belconti resort hotel.

Antalya is a city where tourism is very big business. It has grown massively in the last 30 years, driven largely by Russian tourism, which still continues as Turkey has not blocked Russian flights. The two Russian speakers on the team were able to understand the signage, and the other guests, much better than the rest of us. The drive along the coast to our hotels took us past mile after mile of all-inclusive hotels and golf courses, a place entirely unlike anywhere I'd been before. It included the incredibly tacky Land of Legends hotel, the largest in Turkey, of which we think you'd need a taxi from one end to the other. It is also, however, a historical area. It was part of the Roman empire, and many ruins remain outside the busy city. St Paul visited as he spread Christianity around the Mediterranean, and St Nicholas (of Santa Claus fame) was from this area.

On arrival, Dominic and I ate from the incredibly expansive buffet (one of several restaurants in the hotel) and saw a couple of familiar faces of other leaders. We then picked up the shortlist and settled down to work on the problems separately for a couple of hours. Having got a feel for most of them (it helped that 4 were UK submissions) we discussed them in the hotel lobby with a few other leaders. Nikola, the leader from North Macedonia, was convinced that there was only one set of questions that could be picked; we were skeptical but he was proved right. He told us that he will start a PhD in Bristol this September, supervised by my supervisor's brother, and in a similar area to my own research. We had an early night – tomorrow would be a long day.

## Day 2: Tuesday 9th May

Our main task for the day was to set the paper. We began with a few leaders giving their opinions on the problems in each category, and voting to eliminate certain problems as unsuitable, often because they rely on topics such as analysis which aren't covered until university, or because they are similar to previous problems. As a guest country, the UK leader and observer cannot vote in meetings, but can still give opinions. After an hour of this, it is time to visit the students' hotel for the opening ceremony. On the bus we fill in a survey rating the difficulty and niceness of each problems on a 5-point scale. This is supposed to be more efficient than each leader speaking about every problem, but by the time we waited for them to be put in a spreadsheet and averaged I'm not sure that was the case.

At the ceremony, Dominic and I sit two rows in front of our students, and give them a wave – this is the most we can do, as we have seen the shortlist. The ceremony begins fairly punctually, with an enthusiastic compere introducing speakers in Turkish and English. We had the usual speeches from a representative of the government, and the chairman of BMO 2023, which were shorter than at some past ceremonies. We were then treated to some excellent folk dancing and music, although I'm not sure polystyrene hammers are the traditional props. Rather than the usual procession to the stage, each team was asked to stand up one by one and wave. A group of local singers and dancers closed the ceremony with some more modern music, which led to some of the guides and the Italian team starting a conga line around the room. Our students joined in, though looked more like they were in a chain gang. While the singing was very good, the seven songs they sang were perhaps more than we needed, on a day when we were keen to get back to work.

After transferring back to our hotel and having lunch, the jury reconvened. A few more problems were eliminated or excluded from consideration (a subtle distinction), including some of our favourites and the highest scoring in the beauty contest, but this is not a terrible occurrence, as we can now use them for selection tests in the next year. It was finally time to select the paper. Leaders first proposed pairs of problems for Q3 and Q4, which were then eliminated one by one. Choosing the final questions was very easy, as one pair was supported by all Balkan leaders. During a short break, Dominic and I were consulted on the official English language version, making a couple of minor changes. The other leaders then discussed this, with an eye to translation as certain phrasings are harder to find a close equivalent for in other languages. Once this was approved our work was done – the joy of being an English-speaking country. Other leaders worked on translations, while we had a break and I had a dip in one of the swimming pools, then walked down to the beach. There was a meeting to approve translations, which as a non-polyglot I felt I had little to contribute to, so I spent the time trying to understand the solutions more thoroughly than I'd managed in the little time before the meetings.

## Day 3: Wednesday 10$^{\text{th}}$ May

The competition day. After an early start we were in place at the students' hotel for the exam to start at 9 am. Walking to our meeting room we saw Ina, who told us that 'Neil and Mikaeel are still missing!' It turned out they were already in the exam room. Students are allowed to ask questions of clarification in the first half an hour of the exam. As we had predicted, many questions were of the form 'Can I use the lemma that ...', in particular for the geometry. After some debate about whether 'Yes' implied that the given lemma was true or useful, we decided it was an acceptable response. We could then check in to the hotel, a process which seemed to involve queueing up twice, then waiting for a third member of staff to direct me to my room. The next task was to write markschemes. The coordinators had done a good job at coming up with draft markschemes, and were happy to make the changes that we suggested. However the process happened slowly and before we had fully finished some students were already out and had spoken to their leaders, so we had to leave the markschemes as they were.

I was pleased to meet our students and hear how they had done, with all of them claiming full solutions or close to it on Q1, and some progress on other questions. We thought this was a good result on what had seemed to us a fairly hard paper. I also then met the team's guide, Yasemin, a student at the University of Ankara. Due to the earthquake earlier in the year, students in Turkey had had to study from home for most of 2023. The students showed us to lunch at the beach restaurant, and we got to see their hotel. Being spread over a larger area than the leaders' hotel, it had plenty of space for waterslides, a disco area, stages for performances, multiple swimming pools, a kids area (complete with rabbits) and a large private beach, with camel rides available. This was clearly a very luxurious event! Over lunch Sam discussed his solution to Q4 in more detail. As no leader had been able to find a solution along these lines, we tried to manage expectations, and were pleased to find that his solution with minor tweaks was correct. The afternoon was spent in a relaxed way after the exam, with the students playing volleyball and swimming. We got the scripts just before dinner, where Julia showed us how to make the most of

it by taking 8 desserts.

After dinner it was time for Dominic and me to get to work marking. He took questions 2 and 4, while I took 1 and 3. We also had the Q2 scripts from Turkey A and B teams, as this was a UK submission, and we would have to observe their coordination the following day to ensure fairness. After two hours, we walked to the beach, and found our team on the pull-up bars, and told them how we thought they'd done on questions 1, 2 and 4, to mixed reactions. I carried on working late in the area by the jury room, as unlike my room this had a supply of biscuits, while Dominic had an early night intending to work in the morning. By 2am I was happy with what marks I thought they deserved (rather earlier than last year, when I'd worked until 4am and still had more to do in the morning). I bumped in to the girls in the lobby, and spoke to Julia about her problem 3 script. She was the only one to claim a full solution to this, but unfortunately it didn't work out. The better news was that she was likely to get more marks for problem 1 than I'd initially thought, having done one case and made progress on the other.

## Day 4: Thursday 11$^{\text{th}}$ May

This was the first day of excursions for the students and Ina. They were taken to the Roman ruins at nearby Aspendos, and to see a waterfall which plunged directly into the sea. The stops at a supermarket and while they waited for a bus to bring food were perhaps less interesting.

Meanwhile Dominic and I started with coordination of the geometry at 9am. The coordinators all appeared to be relatively recent contestants, and were very efficient. We always ask for what we think is a fair mark, rather than try to boost our students' scores, which makes it a friendly discussion of maths rather than a debate. In the end we got the mark we'd suggested or better for every script. We had one more complicated meeting, for problem 4, which was postponed until after lunch with the French leader and deputy. Dominic had typed up a corrected version of Sam's script, which the coordinators needed more time to digest. We also had Neil's work, where they were uncertain about whether he had got the mark or not, saying it was worse than the worst one so far to get the mark, but better than the best to not get it. Sam got 8 for his solution, which was the second highest score on this question across the whole competition. Ultimately we finished about 2:30, just about the time our students returned from their trip. While we were pleased with all of them, there was disappointment for some who had got less than they'd hoped for.

I filled the afternoon by going for a run. I picked the option of running on the road rather than soft sand or laps of the football pitch, which meant half an hour of running between hotels on one side and golf courses on the other. I then had an ice cream with Ina, who had waited a while for the fresh Turkish doughnuts to be ready, and eventually got what she wanted. Dominic and I spoke to the Bulgarian and Romanian leaders about the organisation of olympiad maths in our countries, and particularly next

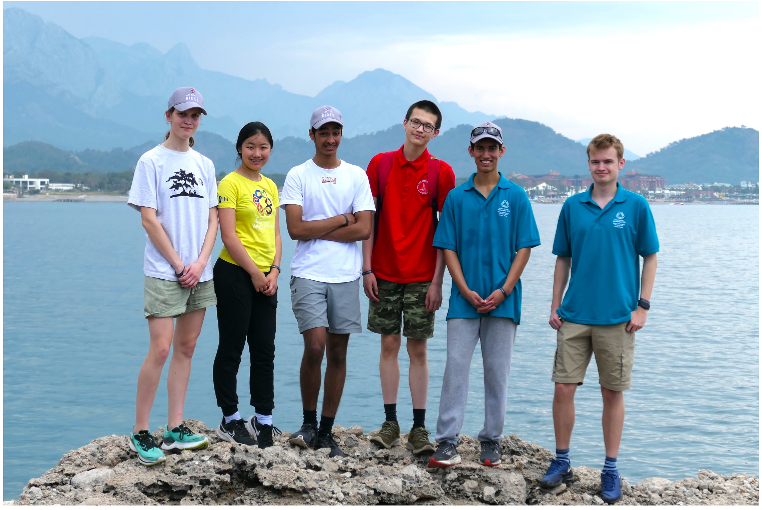
year's Balkan Olympiad in Bulgaria. The leader of North Macedonia was very happy, as it was his country's best results ever, and problem 1 was his proposal. We also heard a report from the French leader and deputy, who had been in a football game, that our boys and especially Anthony had been playing very well.

After dinner (where Dominic found some 'Albanian liver' and reported it to be excellent) was the final jury meeting, with the main aim of choosing the medal cut-offs. It had emerged that lots of students had scored 31, and so we ended up with 31 needed for Silver and 32 for Gold. Congratulations should be given to the Romanian student who got full marks, and the other 4 gold medallists. Romania also came first overall. The cut-offs are based on the official participants, and originally 18 for bronze was approved. However the leader of Uzbekistan had two students on 17, and asked that it be lowered for their benefit, not affecting the official countries. We were happy to support this; a British student had benefitted from a similar decision in 2018. It passed, and so the cut-off was 17. This leaves us with four bronze medals and an honourable mention, a result we can be pleased with. Finally, it was announced that Bulgaria will be the next host.

## Day 5: Friday 12$^{\text{th}}$ May

This was the day of the main excursion, with all teams, leaders and organisers. We were asked to meet at 8:00 for the coach departure. On the way we had a tour guide telling us about the area, its history and economy. Ultimately though, by this point in the week, most of us were rather tired and preferred to have a nap. After about two hours, we reached the harbour on the other side of Antalya, and all boarded a 3-storey boat, with a huge statue of Poseidon at the back. This took us on a pleasant cruise along the coast, with great views of the mountains from the top deck, some of which still had a little snow! While some of the team tried revision at first, a lot of the trip was spent playing cards. By this point our students knew quite a few of the other teams, and seemed to enjoy the time socialising. At one point a disco started on the main deck – while I retreated to the quieter top deck, I am assured that our students did dance.

The destination of the trip was a place called Paradise Island. For an island it looked awfully connected to the mainland, but perhaps there was a gap we couldn't see. Here we disembarked from a jetty that



was comically small for such a ship, and were told we had an hour to walk around. Seeing the dark clouds we decided to stay near the boat and just take some photos. The guides seemed to agree and stopped anyone going too far, then after 20 minutes got us all back on the ship. All, that is, except the North Macedonian team, who had managed to go for more of a walk. Numerous announcements over the ship's PA system were unsurprisingly ineffective at getting them to return, but we were on our way again soon enough. At this point the boat looked alarmingly on fire, but this turned out to be a barbecue of lamb, chicken and delicious fish, which we all enjoyed on the return journey.

The return coach journey was another sleepy affair. The tour guide predicted this, as we had 'obtained many calories' and let us sleep. This afternoon saw the only rain of our time in Turkey, and some river beds that had been dry on the way to the boat were now in full flow. After snoozing some of our students tried to digest the solution to problem 3, especially the unfamiliar Schur's Theorem. We were driven to a petrol station, and offered a five minute break to use the toilets. After 30 minutes, we eventually left and drove two minutes to the Expo centre, where the closing ceremony would take place. Here we had time to change (with Mikaeel having to borrow my polo shirt) and have snacks – we had well over an hour to wait until the dignitaries would be there and ready. As we took our seats, the closing ceremony began with an advert for Turkey, played 14 times consecutively to increasing amounts of applause. Once it started properly, the ceremony was very well-done, with short speeches from the appropriate organisers and representatives of sponsors, then a few minutes of footage from the week, including of myself and Dominic in a coordination meeting. The medals were presented, then a photo of all contestants was taken on the stage.

Dinner was held nearby, at what seemed to be more usually a wedding venue. There was a moment of comedy on the way, as the first coach tried and failed to enter the grounds via an archway which was too small – fortunately our driver did not attempt the same, and we walked up the hill to arrive first. From past experience, we knew it was safest to take the table furthest from the band. A long dinner of many courses followed, and it was nearly 11 by the time we got back to the hotel. Certificates were given to all, in red velvet folders, and we said goodnight to the students, who headed to the disco.

## Day 6: Saturday 13$^{\text{th}}$ May

Our coach was at 11:15, so Dominic and I both found time for a final swim before departure. Our journey home was a bit chaotic. We, with the Greeks and some of the Turks, were on a bus with far too little luggage space, so there were suitcases piled on seats an in the aisle. Having read Geoff's report, we were wary of the distinction between terminals at Antalya airport, and were prepared when we arrived at the domestic terminal. Here the driver unloaded all of the suitcases from the luggage compartment, except the Greek observer's. After a bit of a commotion, this was found in a different luggage compartment. We enlisted the help of a Turkish student to explain that we needed to go to Terminal 2, and reloaded our suitcases. Neil realised in this time that his wallet was missing, last seen in his hotel room, so when we could get to the airport wifi we got in touch with Neil's mother, who could attempt to use an airtag to find it, as well as our guide and the organisers. Unfortunately this was to no avail.

Antalya airport seems to require a lot of admin; we first went through a security check, then dropped off our bags and had to re-print our boarding passes, then had boarding passes checked, then passports stamped, another round of security, passports and boarding passes checked twice before getting on the plane, then boarding passes once more at the door. Perhaps this explains why we were dropped off 3 hours early, and still felt short of time for lunch.

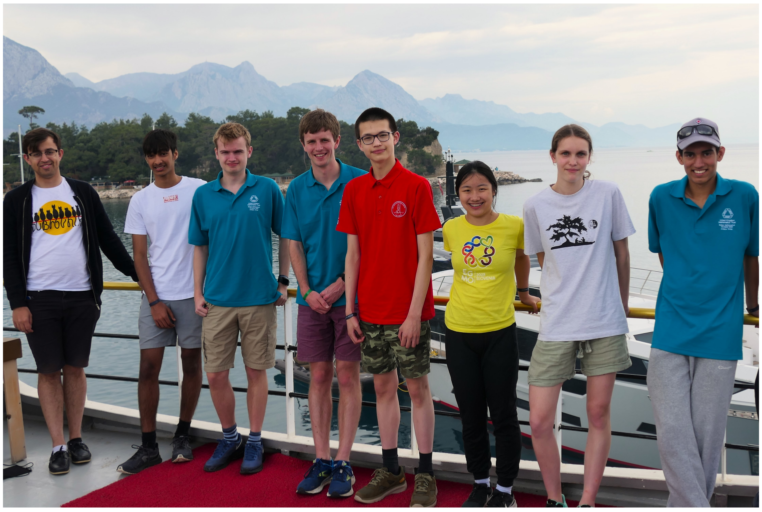
On arrival, Ina took Elsa through the non-UK citizens line, behind a full plane from Brazil. It took the best part of an hour with little progress before Ina spoke to staff who let them through the UK queue. The rest of us were through in the arrivals area, and over half an hour Neil and Mikaeel were picked up, and Dominic, Julia and Anthony left to get trains and buses. Sam's parents were stuck on the M25 for 3 hours, but kept us informed of their ETA which was much appreciated. Elsa was also picked up, and Ina and I caught the train up to London. I made it home 12 hours after leaving the hotel, but others were less lucky. Anthony's train to Exeter, the last of the night, had been cancelled, and he had to book in to a hotel.

## Conclusion

This was a very enjoyable trip, and a successful one too. There are many people who have put a lot of work in to making this happen. I'd like to thank

- Azer Kerimov, the chair of the Olympiad, and all working with him to put on this event. The volunteers I met were universally friendly and helpful, and from our perspective the efficiency of the co-ordinators was much appreciated
- Yasemin Seker, our team's guide, who kept us informed of what was going on, and helped with language issues
- Natalie Ruecroft, and all at UKMT who made our participation possible
- Dominic Yeo, for all he does for the Olympiad teams, but especially for the support and experience he brings helping me in the mathematical matters of the competition
- Ina Hughes, who ensured the team were safe, well-fed, rested and in the right places for the exam and other events, and whose company we all enjoyed

and everyone else who helped with running the Olympiad and preparing the students for it. Finally I'd like to thank the team for their company. They were excellent representatives of their country, and can all be proud of how well they've done.



# Appendix

## Alternative solution to Q4 (UNK 2, edited by DJY)

We prove that $k = 592$ is possible. Let $A \subset [2023]$ be Alice's set, and write $b_1 > b_2 > b_3 > \ldots$ for the elements of $[2023] \setminus A$ (which are available for Bob to select), and

$$\alpha = \sum_{a \in A} a, \quad \beta_j = \sum_{i=1}^{j} b_i.$$

We note that $\sum_{b \in [2023] \setminus A} b \geq \alpha$. Now, let $i$ be the minimal index such that $\delta = \alpha - \beta_i < b_i$. Note that this is not always equivalent to saying 'let $i$ be the largest index such that $\beta_i \leq \alpha$' (which would be the standard setup in a greedy algorithm).

If $\delta = 0$ or $\delta \in A^c$, then we are done, by taking $B = \{b_1, \ldots, b_i\}$ or $B = \{b_1, \ldots, b_i, \delta\}$, respectively. We also note that if $b_i = 1$, then either $\alpha = \beta_i$ or $\alpha$ is greater than half the total sum (which is not possible for $k = 592$). Similarly, if $b_i = 2$, then the only remaining case is $1 \in A$ for which $\alpha$ is again greater than half the total sum.

So in the following we may assume $b_i \geq 3$. We assume (for contradiction) that $\delta \in A$, and no suitable $B$ exists. In particular, this implies

- In every pair $(1, \delta - 1), (2, \delta - 2), \ldots \left(\lfloor \frac{\delta-1}{2} \rfloor, \lceil \frac{\delta+1}{2} \rceil\right)$, at least one of the two elements is in $A$.
- In every pair $(\delta + 1, b_i - 1), (\delta + 2, b_i - 2), \ldots, (\delta + \lfloor \frac{b_i - \delta - 1}{2} \rfloor, b_i - \lfloor \frac{b_i - \delta - 2}{2} \rfloor)$, at least one of the two elements is in $A$, as otherwise, we may take $B = \{b_1, \ldots, b_{i-1}, \delta + m, b_i - m\}$, which has sum $b_1 + \ldots + b_i + \delta = \alpha$.
- Note that we discuss $\lfloor \frac{\delta-1}{2} \rfloor + \lfloor \frac{b_i - \delta - 1}{2} \rfloor$ pairs in total, and they are all disjoint.

Note that $\lfloor \frac{\delta-1}{2} \rfloor + \lfloor \frac{b_i-\delta-1}{2} \rfloor \geq \frac{\delta-2}{2} + \frac{b_i-\delta-2}{2} = \frac{b_i}{2} - 2$. Finally, recalling that $\delta \in A$, we conclude that at least $\lceil \frac{b_i}{2} \rceil - 1$ of the elements of $\{1, 2, \ldots, b_i - 1\}$ are in $A$.

- so at most $593 - \lceil \frac{b_i}{2} \rceil$ of the elements of $\{b_i + 1, \ldots, 2023\}$ are in $A$;
- so $i \geq (2024 - b_i) - (593 - \lceil \frac{b_i}{2} \rceil) = 1431 - \lfloor \frac{b_i}{2} \rfloor$.
- In the other direction, we have shown that $\frac{b_i}{2} - 2 \leq 592$, ie $b_i \leq 1188$.

Consequently, we can bound $\beta_i$ as

$$\begin{aligned} \beta_i \geq b_i + (b_i + 1) + \ldots + (b_i + (i-1)) &\geq b_i + (b_i + 1) + \ldots + \left(b_i + 1430 - \lfloor \tfrac{b_i}{2} \rfloor\right) \\ &= T\left(b_i + 1430 - \lfloor \tfrac{b_i}{2} \rfloor\right) - T(b_i - 1), \end{aligned} \tag{1}$$

where $T(m) = 1 + \ldots + m = \frac{m(m+1)}{2}$ is the $m$th triangle number. We want to claim that over the range $3 \leq b_i \leq 1188$, the quantity in (1) is $> \frac{2023 \times 2024}{4}$. For $b_i = 3, 4$ we check this manually! (*) To avoid integer-valuation issues, we set $\bar{T}(x) = \frac{x(x+1)}{2}$, and rewrite (1) as

$$\beta_i \geq \bar{T}\left(b_i + 1430 - \tfrac{b_i}{2}\right) - \bar{T}(b_i - 1).$$

We have $\bar{T}' \geq 0$ on the range $x \in [4, 1188]$, so for this range minimum is obtained at $x = 4$:

$$\beta_i \geq \bar{T}(1432) - \bar{T}(3) = \frac{1432 \times 1433}{2} - 6 > \frac{2023 \times 2024}{4},$$

by the same manual calculation as (*). In all cases, we find that $\beta_i$ is greater than half the total sum, a contradiction.