IMO 2018, Romania - UK Report

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The UK Maths Trust[2] organises competitions, mentoring and other enrichment activities for talented and enthusiastic school-aged mathematicians. One strand is a training programme for the country's top young problem-solvers to introduce them to challenging material, enjoyable in their own right, but also the focus of international competitions.

The International Mathematical Olympiad is the original and most prestigious such event, now in its 59th edition. About a hundred countries send teams of up to six contestants, and this year I organised some of the preparation for the UK team to take part at the IMO in Cluj-Napoca, Romania. The team leader was Geoff Smith, who is also the IMO Board President. The academic team for our pre-IMO training in Budapest was enhanced by the addition of Freddie Illingworth, who participated in IMO 2014, and is about to start a PhD in combinatorics.

This report includes a lengthy introduction to the problems of IMO 2018, and a brief discussion of the results within the context of the UK training progamme. Much more detail about these results, as well as the perspective from the leaders' site, can be found in Geoff's complementary report. This report also includes a diary describing somewhat frivolously our mathematical and cultural experiences during the final training camp, and the competition itself. A version with pictures will be available at the olympiad page on my blog[3] at some point. Thank you to the UK students who contributed various passages, and politely corrected several mistakes, ranging from details of the excursions, to the case a_1 composite in Question 5.

Commentary on the problems of IMO 2018

I hope these discussions of aspects of the problems may be interesting and useful to a wide range to readers. I also hope that it's clear that trying the problems yourself is also interesting and useful, especially to aspiring students, and that trying the problems yourself will be less valuable after reading these commentaries, even though they are not intended as formal solutions.

Problem 1

Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

There are many entry routes into this problem. Under any of them, you have to have a plan for using the length condition $AD = AE$, and it's also useful to have in mind that 'parallel to

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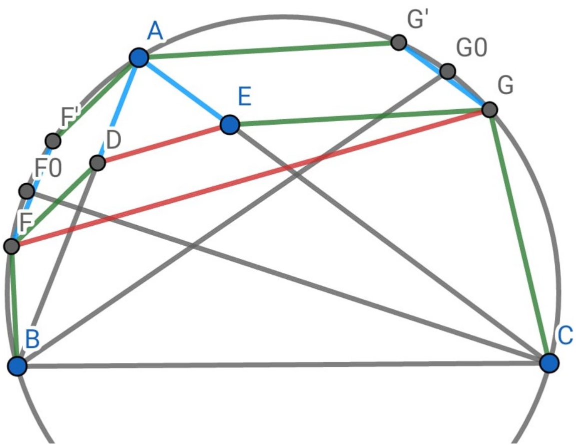
DE' is certainly equivalent to 'perpendicular to the A-angle bisector'. The line DE itself may not be crucial.

My own solution looks rather long, but is hopefully educational. The two most motivating comments are:

- Consider the case $D = E = A$. Then F and G are the arc-midpoints of minor arcs AB and AC. Can I prove this specific simple case? It's a very useful observation that the AB arc-midpoint lies on the circumcircle Γ, the perpendicular bisector of AB, and the C-angle bisector. Why not try yourself to see if you can directly calculate the angle between DE and AI using this?
- There are three sets of equal lengths, namely $AD = AE$, $BF = DF$, $CG = EG$. The conclusion is about parallel lines. The whole diagram lives in a circle, which allows us to move angles around easily. So it's likely to be possible to find some useful parallelograms or isosceles trapezia. (Note that a cyclic trapezium is always isosceles.)

Thanks to Aron for the following diagram:

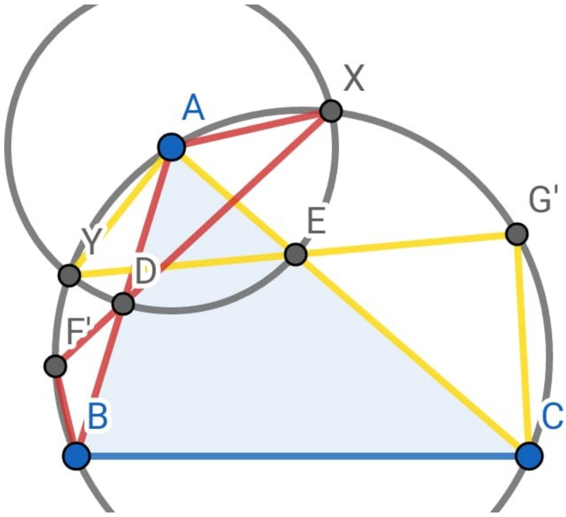


Let F_0 be the arc-midpoint of AB, and let F' be the reflection of F in the perpendicular bisector of AB. Define G_0, G' similarly, though for now we'll just attack one side of the diagram. The goal is to prove that $ADFF'$ is a parallelogram. But by construction, AF' subtends an equal angle to BF, and so $ABFF'$ is an isosceles trapezium. Since D lies on AB and $\triangle DBF$ is isosceles, we obtain that $ADFF'$ is indeed a parallelogram. This is useful because we can do the same thing on the other side, to deduce $FF' = AD = AE = G'G$.

Our overall goal is to show that $FG \parallel F_0G_0$ since we already know (if we read the first bullet point above) that $F_0G_0 \perp AI$. So it suffices to show that FF_0G_0G is an isosceles trapezium, which requires $FF_0 = GG_0$. We already have a similar length condition, but if we convert both into statements about subtended angles, this equivalence becomes clear. We have $\angle FCF' = \angle G'BG$ from $FF' = G'G$. But then by construction of F', G', the angles $\angle F_0CF$, $\angle F'CF_0$, $\angle G'BG_0$, $\angle G_0BG$ are all equal, and so in particular we can read off $FF_0 = GG_0$, and confirm we have the isosceles trapezium FF_0G_0G which was our goal. $\square$

Alternative solutions: Proceeding directly with a trigonometry is clearly possible if you plan your route through the calculation carefully. If your $\mathbb{C}$omplex geometry toolkit is rich enough, then since this diagram lives on a circle, you can join Sam and Tom in deploying such methods.

Aron says he was motivated by the thought of drawing the diagram in Geogebra[4], which would require the circle centred at A through D and E.



Aron writes: We capture the length equality $AD = AE$ by drawing the circle ω centered at A through D and E. Turning into a condition about points lying on a circle might not turn out to be important or useful, but it's often helpful trying to define points in different ways.

The next step is to conjecture that FD and GE each passes through one of the common points of the circles Γ and ω. Again, an accurate diagram helped a lot! Call these intersections X and Y as shown. It's not immediately clear that proving this will be any simpler than the original problem. However, we observe that if the conjecture is true, then since $XYDE$ and $XYFG$ are cyclic, both DE and FG are antiparallel to line XY with respect to XD and YE, and so DE and FG are themselves parallel.

Trying to prove that XDF and YEG are collinear directly isn't easy. To work around this, define the second intersection of XD with the circumcircle as F'. Then XAD and $BF'D$ are similar since XF' and AB are chords of Γ that intersect at D. But $AX = AD$, so this means $FB = FD$ too. Noting that F' lies on minor arc AB, we find that F' really is F, and so XDF are collinear. Likewise YEG are collinear, so we are done. $\square$

Problem 2

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_{n+2}$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}, \tag{1}$$

for $i = 1, 2, \ldots, n$.

Unsurprisingly, since otherwise the question would be not so interesting, you can check it's not possible to have an example with $a_1 = a_2 = \ldots = a_n$. Other small cases are useful, though. In particular, $n = 3$, which you can handle through a couple of quadratic equations, to obtain a potential solution $\{2, -1, -1\}$, and its cyclic permutations. Indeed, the cyclic nature

[4]Geogebra is free software for drawing and analysing diagrams such as these two. It supports export to TiKZ for LaTeX and so on. Outside contests themselves, it's an excellent resource for checking whether conjectures (collinearity etc) are true. It is now available for a mobile app for all platforms, which was being used heavily by this year's team, and increasingly by their deputy leader after two weeks of near-constant exposure...

of the statement (where each new term depends precisely on the previous two), indicates that $(2,-1,-1,2,-1,-1,2,-1,-1,\dots)$ is a valid solution whenever $3 \mid n$.

Note though, that this pattern will not 'join up correctly' if $3 \nmid n$. Again, how long you spend hunting alternative examples when $3 \nmid n$, versus deciding to start proving it isn't possible, is a matter of taste, and the strategies are not all that different. If your attempts at a construction are failing in a particular way which you can define, that's often the basis of a proof.

It turns out that the most standard approach is studying which terms in a valid sequence could be positive or negative, and to derive a contradiction if the pattern is not $(+,-,-,+,-,-,\dots)$. But checking my notes, this was certainly not the only option I considered in the deputy leaders' room. Among other things, I was struck by the fact that one way to characterise the (apparently only) solution is $(2\cos(0), 2\cos(\frac{2\pi}{3}), 2\cos(\frac{4\pi}{3}), 2\cos(\frac{6\pi}{3}),\dots)$ etc. Anyway, it didn't seem impossible that (1) could be converted into some suitable double-angle formula, and this might be revealing. Well, I failed utterly to achieve this - perhaps others had more success?

Most standard solutions would have a short series of steps like:

- Claim 1: impossible to have two consecutive positive terms, or a single zero.
- Claim 2: impossible to have three consecutive negative terms.
- Claim 3: impossible to alternate positive and negative (only relevant if n is even).

Each of these steps should be proved. Claim 3 requires more work than the others. It's useful to focus on the negative terms and recall that they cycle. The results show that the three British students who did not complete their solutions all earned valuable marks for *stating and proving* partial results of this kind. It's essential to do this properly, and make it look like you know what you are proving. Not just for scraping up IMO marks, but because being organised about partial results increases the chance you'll see how to combine them for a full result.

I finished by showing that the pattern $(-,-,+,-,+)$ could not appear, though there were many alternatives. You can sequentially find actual bounds on each of the last three terms. For example, the first positive term must be at least 1, and in particular the product of the final two terms is at least -1. Which means that the next term in the sequence would be positive, contradicting Claim 1. $\square$

Unlike on more conceptual questions, it's worth emphasising that small mistakes are very dangerous here, because it really won't be clear that an argument that shows $b < -1$ then uses $b > -1$ can instantly be fixed. There is a magically short solution too. Avoid this footnote[5] if you don't want to know what it is.

[5] Try to find an expression for a_i^2, and then sum this appropriately over all i, to obtain $\sum_i (a_i - a_{i+3})^2 = 0$. Then don't feel bad that you didn't see this earlier. You can afford to be less creative about the summing if you know the *rearrangement inequality*.

Problem 3

An 'anti-Pascal triangle' is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 *to* 10*.*

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 *rows which contains every integer from* 1 *to* $1 + 2 + \ldots + 2018$*?*

My immediate reaction was 'surely not', and this was reinforced by unsuccessful attempts to construct one. We can see from the statement that $n = 4$ is not so instructive for a negative answer, and I didn't try other small examples. But consider the following. Let d be the difference of two numbers a, b. Then the minimum of these three numbers is at most half the size of the maximum of the three. (*) This felt like a strong observation. How are you going to fit in the larger half of the required set of numbers? (In what follows, write $T_n = 1 + 2 + \ldots + n$.)

I found it challenging to turn this into a full argument, but it motivated looking for a chain of increasing values from the top to the base of the triangle. Based on (*) we might worry that such a chain should increase *too fast*. That is, too fast for the final value to be at most T_n.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & \mathbf{6} & & \\ & 5 & & \mathbf{7} & & 1 & \\ 8 & & 3 & & \mathbf{10} & & 9 \end{array}$$

In the above diagram, the bold numbers represent $a_1, \ldots, a_n$, a chain of values starting from the apex, where given a_i, we choose a_{i+1} to be the larger of the two numbers whose different is a_i. The circled numbers represent $d_i := a_i - a_{i-1}$ for $i \geq 2$, and $d_1 = a_1$. In the given example, we have $a_4 = 10 = T_4$, and $\{d_1, d_2, d_3, d_4\} = \{1, 2, 3, 4\}$. This is not a coincidence. Indeed, in general, we have $a_n = d_1 + d_2 + \ldots + d_n$, but the values of the d_is are *distinct*, so we must actually have

$$a_n \geq 1 + 2 + \ldots + n = T_n.$$

But of course, $a_n \leq T_n$ since that's the maximum integer we are putting in the triangle, so the only option is that $a_n = T_n$, and $\{d_1, \ldots, d_n\}$ forms a permutation of $\{1, 2, \ldots, n\}$.

This is a very strong condition on the triangle indeed. In particular, we need to use all the integers $\{1, 2, \ldots, n\}$ on or adjacent to this *maximal chain*. But we can find other large chains. Suppose we have another chain $b_1 \leq b_2 \leq \ldots \leq b_\ell$ that's completely separate from the maximal chain, so not even the adjacent numbers (ie the circled ones) overlap. (One should define this a bit more formally, though diagrams make life easier.) Consider, as before, the differences $e_i := b_i - b_{i-1}$ and $e_1 = b_1$. Then, as before, we have

$$b_\ell = e_1 + e_2 + \ldots + e_\ell \geq (n + 1) + (n + 2) + \ldots + (n + \ell), \tag{2}$$

since the e_is are disjoint, and different to all the d_is. Now, by considering where the maximal chain meets the base, we can exhibit an appropriate chain for which $\ell \sim n/2$. For example, I think $\ell \geq \lfloor n/2 \rfloor - 5$ is definitely fine. But now if we substitute this into (2), we can show that $b_\ell > n^2/2 > T_n$, so long as n is large enough that this floor and the -5 doesn't make a big impact (which definitely *does* affect the small case $n = 4$). □

Agnijo did not need to find a second chain as the final part of his solution, which returns to the territory of (*).

Agnijo writes: define a number to be *small* if it's $\leq n$ and *large* if $\geq T_{n-1}$. If we have a large number in any row other than the bottom row, the two numbers beneath it must be a small number and another large number. Since there is only one small number in each row, there are at most two large numbers in each row other than the bottom row.

We can also consider possible values for the maximum number in each row, and show that this can only be a large number when the row is near the bottom. In particular, if $T_k > n$, there are no large numbers in or above row $n - k$.

In this way, we obtain that almost all large numbers must be in the bottom row. Precisely, at least $n - 2k + 3$ such numbers, as there are $n + 1$ large numbers and at most $2k - 2$ outside the bottom row. For large n (and $n = 2018$ certainly works), this is much greater than $n/2$ so we can find at least two pairs of adjacent large numbers in the bottom row. But if we have such a pair, the number directly above the pair must be small, and there is only one small number in the second row. Thus we have found a contradiction.

Problem 4

A 'site' is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 *sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. One her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.*

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

A question of this form requires providing separate descriptions of *two* strategies. Firstly, you have to give a strategy for Amy which enables her to place K stones as requested. Then, you have to give a strategy for Ben which enables him to prevent Amy placing $K + 1$ stones. You don't know a priori 1) what K is; 2) whether the two strategies will be intimately related; 3) which of these will be more taxing either to find or to prove. However, it is reasonable to guess that there might exist *simple* descriptions of strategies for both players, because it's been chosen as a problem on the IMO. If you have reduced to a very large number of cases, or have to study five moves into the future, this may well work, but you should keep your eyes peeled for more appealing characterisations, which might be easier to handle in proofs.

So if you are visualising the problem on a 20×20 grid, then Amy is banned from placing two red stones a *knight's move*[6] apart. The language of chess encourages us to think about colouring the grid in some fashion. There are many ways to colour a grid, but we might notice that the usual *chessboard colouring* has the property that a knight's move involves two cells with *different* colours.

Amy can guarantee that she'll never face knight's move obstacles if she only uses one colour (say black) for her red stones. There are 200 black cells, so even if Ben also occupies lots of black cells on his turns, Amy can guarantee filling at least 100 black cells. If you write this more formally, you have given a strategy for Amy to achieve $K \geq 100$, and proved that it works.

What next? Well, you don't yet know that $K = 101$ is not possible. Maybe Amy can use half the black cells and then also use a white cell too? Can Ben definitely prevent this by his choice of black cells? Maybe there are other totally unrelated ways to get $K = 101$? How long you spend thinking about this is a matter of taste, but eventually we decide that if there exists a strategy guaranteeing $K = 101$ then we're not smart enough to find it, so we should conjecture 100 as the bound and try to find a strategy for Ben which *prevents* Amy achieving 101.

Our exploration is still underpinned by the moral that there must exist a strategy for Ben with a simple description (remembering that 'simple' and 'easy to find' are different). Some ideas:

- It would be great if we could pair up the cells suitably so that a strategy for Ben is "whenever Amy plays in cell C, Ben plays in cell C's partner." That way, there'll never be any ambiguity about whether the cell Ben wants to play in is actually available.
- If Ben always plays far away from Amy (eg in the 'opposite' cell) then Amy can easily get 101. Indeed she can get 180 by using all the blacks in one half, and all the whites in the other half, avoiding the two rows at the boundary of the halves[7]. So we probably need Ben to choose a cell fairly close to Amy's previous cell.
- Perhaps some smaller examples will help? A 2×2 grid has no knight's moves at all. The construction we already have for Amy is a bit messier for 3×3 and 5×5, since there are different numbers of black and white cells, so these are less likely to be useful than 4×4.
- If we can give a strategy for Ben to prevent $K = 5$ on a 4×4 grid, then we can specify a strategy for Ben to prevent $K = 101$ on the original grid. All we have to do is split it up into 25 small grids, and insist that Ben plays in the same small grid as Amy, and follows the appropriate strategy for that small grid.

That's enough motivation. If you play with the 4×4 case, it will start to seem clear that Ben can prevent 5 red stones appearing. The same moral about simple strategies applies, yet now the idea of Ben playing in the *opposite cell to Amy* is a plausible option. With this strategy, each move by Amy affects the following cells: a) one cell now filled by a red stone; b) one cell now filled by Ben's blue stone; c) two further cells can't be accessed by Amy in the future. So each move by Amy reduces her future option set by 4. It remains to prove that these are disjoint, for example by, colouring the 4×4 grid with four colours. Each colour represents a rhombus with side length $\sqrt{5}$, or a four-cycle tour of knight's moves. We argue that with Ben's strategy, Amy can fill at most one cell of each colour, leading to $K \leq 4$ in the 4×4 grid, and $K \leq 100$ in the original 200×200 grid when Ben copies this strategy for each 4×4 subgrid. □

[6]Indeed, the problem proposers used this language as part of their original formulation.
[7]This was the first example I could think of. Perhaps she can do even better?

Problem 5

Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \tag{3}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

I'll use *eventually* as a shorthand for 'whenever n is large enough', though the exact threshold may be different in different uses. If you study (3) for both n and $n+1$, and consider their difference, you find that

$$\frac{a_n}{a_{n+1}} + \frac{a_{n+1} - a_n}{a_1} \tag{4}$$

must eventually be an integer. This problem statement is *not* cyclic in the variables $\{a_1, a_2, \ldots, a_n\}$, and we can see from (4) that a_1 has a special role, so let's denote $a_1 = K$. It's worthwhile to study some special cases of K.

$\mathbf{K = 1}$: In this case, $\frac{a_{n+1}-a_n}{K}$ is always an integer, so we must have $\frac{a_n}{a_{n+1}} \in \mathbb{Z}$ eventually too. But this means that $a_{n+1} \leq a_n$ eventually. So we have an infinite non-increasing sequence of positive integers, which must therefore be eventually constant, as required.

$\mathbf{K = p}$, **prime :** Eventually, from (4) we have

$$\begin{cases} \dfrac{a_n}{a_{n+1}} = \dfrac{\ell_n}{p}, & (5) \\ a_n - a_{n+1} \equiv \ell_n \mod p, & (6) \end{cases}$$

for some integer sequence (ℓ_n). Observe that if $p \mid a_n$, then from (5), $p \mid a_{n+1}$ unless $p^2 \mid \ell_n$. But if $p^2 \mid \ell_n$ and $p \nmid a_{n+1}$, this contradicts (6). So in fact $p \mid a_{n+1}$ and thus by (6), $p \mid \ell_n$ in this case. So the two options are that

- for some $M \geq N$, $p \mid a_M$, in which case we've argued that $p \mid a_n$ for all $n > M$. We've therefore argued that eventually $\ell_n \geq p$, so eventually $a_{n+1} \leq a_n$ by (5), and we can conclude as before;
- Or eventually $p \nmid a_n$, which forces $p \mid \ell_n$ eventually from (5); and so again we have $\ell_n \geq p$, leading to $a_{n+1} \leq a_n$ eventually, and we can conclude as before.

I think much of the insight is contained in these cases, though the majority of any solution to the original problem will be devoted to handling the different primes p dividing K separately, and the possibility that $p^2 \mid K$. In the full solution which follows, Aron uses the language of the *valuation* $v_p(n)$, which records the exponent[8] of the largest power of p dividing an integer n.

Aron completes the solution: As Dominic has written, the first step is to consider the difference $\frac{a_n}{a_{n+1}} + \frac{a_{n+1}-a_n}{a_1}$, which reduces the condition from n variables to two variables and one constant, namely a_1, which should make it much easier to analyse.

This difference must be an integer. Writing it as a single fraction we have

$$a_{n+1}a_1 \mid a_n a_1 + a_{n+1}^2 - a_n a_{n+1}.$$

[8] So, for example, when $\{2, p, q\}$ are all different primes, we have $v_p(4p^3q^2) = 3$.

It is possible to work with this, but note that subtracting $a_{n+1}a_1$ gives

$$a_{n+1}a_1 \mid (a_{n+1} - a_n)(a_{n+1} - a_1). \tag{7}$$

The motivation for how to proceed from here is that (7) immediately implies a_{n+1} divides $a_n a_1$. So if $a_1 = 1$, we would be done since we have a non-increasing sequence of positive integers, which must eventually be constant.

This motivates looking at prime divisors p of a_1, since it would suffice to show that p-adic valuation of $\gcd(a_1, a_n)$ is eventually constant for every such p (since there are only finitely many). From this it follows that $\gcd(a_1, a_n)$ is eventually constant. Since (7) is homogenous, we can eventually divide a_1, a_n and a_{n+1} by this greatest common divisor. Then we would still have a_{n+1} dividing $a_1 a_n$, but now a_1 is coprime to both a_n and a_{n+1}, so a_{n+1} divides a_n for large n, which is what we want.

Going back to (7), we can phrase this in terms of the p-adic valuation of both sides:

$$v_p(a_{n+1}a_1) \leq v_p((a_{n+1} - a_n)(a_{n+1} - a_1)) \tag{8}$$

Now we make use of two facts about v_p, namely $v_p(ab) = v_p(a) + v_p(b)$, and

$$v_p(a - b) = \min\left(v_p(a), v_p(b)\right), \quad \text{unless } v_p(a) = v_p(b).$$

If for any $n \geq N$, we have $v_p(a_n) \geq v_p(a_1)$, then suppose $v_p(a_{n+1}) < v_p(a_1)$. But then from (8),

$$v_p(a_{n+1}) + v_p(a_1) \leq 2v_p(a_{n+1}),$$

which is a contradiction. So in fact $v_p(a_{n+1}) \geq v_p(a_1)$ too. So if, after N, the p-adic valuation of a_n is ever at least that of a_1, then by induction it remains so for all a_m, with $m \geq n$, in which case $v_p(\gcd(a_n, a_1)) = v_p(a_1)$ eventually.

Otherwise, if the p-adic valuation of a_n is eventually less than a_1 (*), either this valuation is eventually constant, or it is not. If it is not eventually constant, then there exists m beyond the threshold for (*) such that $v_p(a_m)$ is less than $v_p(a_{m+1})$. Now from (8) we have

$$v_p(a_{m+1}) + v_p(a_1) \leq v_p(a_m) + v_p(a_1), \quad \text{unless } v_p(a_1) = v_p(a_{m+1}).$$

So this forces $v_p(a_{m+1}) = v_p(a_1)$, which contradicts (*).

In the second case, if $v_p(a_n)$ is eventually constant and less than $v_p(a_1)$, then $v_p(\gcd(a_n, a_1))$ is also eventually constant, and we are done. □

Problem 6

A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA. \tag{9}$$

Prove that $\angle BXA + \angle DXC = 180°$.

It turned out that this question was open to vast number of modes of attack, some of which benefited from more knowledge than others. Here, Agnijo gives an outline of his solution from the competition. We will brush over questions of existence and uniqueness of points (for which the convexity and interior assumption in the statement is useful), and anything to do with angle direction, but these would be more relevant than usual in lots of solutions.

It's a recurring theme of olympiad geometry that angle conditions are often the same as circle conditions. At more junior levels, the deduction is normally the other way round: given a circle condition, you decide which circle theorem will release the most useful angle conditions to work with[9]. At IMO hard level, it's more likely to be the other way round where, like here[10], you are given an awkward angle condition, and want to reduce to a more pliable statement. You just have to play around, and notice that the circle $\odot BXD$ generates opposite supplementary angles when it meets AD again, so long as this intersection point also lies on BC. So taking $E = AB \cap CD$ and $F = AD \cap BC$, (9) describes X as the[11] intersection of $\odot ACE$ and $\odot BDF$.

Agnijo writes: Now, we can find $\angle CXD - \angle AXB$. With reference to $\triangle AXB$ and $\triangle CXD$, we have $\angle CXD - \angle AXB = \angle ABX - \angle CDX = \hat{B} - \hat{D}$. (*)

Meanwhile, suppose $\odot AXB, \odot CXD$ meet at Y. A short angle-chasing argument reveals that Y lies on BD. Furthermore, since Y lies on both $\odot AXD, \odot CXD$, the angles $\angle AXB$ and $\angle AYB$ are equal, and similarly $\angle CXD = \angle CYD$. Thus it suffices to prove the required angle condition for Y instead of X.

But we can adapt (*), deriving $\angle CYD - \angle AYB = \hat{B} - \hat{D}$. Since Y is on the diagonal BD, we can rewrite this as $\angle AYC = 180 - (\hat{B} - \hat{D})$. At this point, it suffices to prove that $\angle AYB = \angle BYC$. In other words, that BD is the angle bisector of AYC.

Now we use the condition that $AB \cdot CD = BC \cdot DA$, in the form $\frac{AB}{AD} = \frac{BC}{DC}$, since this is reminiscent of the angle bisector theorem. Indeed, it tells us that there is a point Z on segment BD such that the angle bisectors of A and C meet at Z. By construction, $\odot AZC$ is an Apollonian circle[12] of $\{B, D\}$. Using the angle bisector property, we obtain $180 - \angle AYC = \hat{B} - \hat{D} = 360 - 2\angle AZC$.

This gives us all the ingredients required to show that BD is the angle bisector of AYC. Agnijo used a short but rather clever Euclidean argument involving an isosceles trapezium, but there are other methods too. $\square$

Indeed, one can also come up with alternate characterisations of the conclusion. For example, if you glue a similar copy of $\triangle BCX$ outside side AD as $\triangle ADZ$, then the required conlusion is that $AXDZ$ is cyclic, but it already has parallel sides, and so it suffices to prove that it is an isosceles trapezium. Or alternatively, that $AX = DY$, so long as one has some other reason why it can't be a parallelogram.

This reduces to showing $\frac{BX}{DX} = \frac{AB}{CD}$ etc, though this reduction could, equally, have been derived from the sine rule. It turns out that inversion at X is a good way to proceed, as the image of $ABCD$ is similar to the original figure. Alternatively, especially after the cyclic recharacterisation of the given angle conditions, inversion at A is nice since there are now many relevant lines and circles, and the length condition given becomes a statement about an isosceles triangle in the inverted diagram (which might, according to Andrew, be well-known to some, but was not well-known to me, so proving this and finishing still required significant work).

[9] Eg BMO2 2018 Q1 - it's worth explicitly stating that for this question, you really gain nothing by *drawing the tangent circle* through the three relevant points. This is nothing more than a problem composition device for introducing an angle relation in a more appealing way. `https://bmos.ukmt.org.uk/home/bmo2-2018.pdf`

[10] If you want to try another example, see if you can convert the statement of IMO 2014 Q3 similarly.

[11] Modulo uniqueness, as mentioned.

[12] In this context, we mean the locus of points W such that the angle bisector of $\angle BWD$ passes through Z. By construction, A and C have this property!

Overall comments

The questions on this IMO were all appealing and produced a set of results that discriminated appropriately to maintain a meaningful sense of competition over actual solutions. The medal cutoffs in Cluj corresponded to roughly $(2, 3.5, 4.5)$ full solutions. This is much more satisfactory than the $(2, 2.5, 3.5)$ that turned out to be the case in 2015 and 2017. Quite simply, the medium questions in those two years were unwelcomely difficult, and so for the better-prepared half of the students, the competition was reduced to a fight for $\sim 4/28$ partial marks on the questions which they couldn't solve, which was a mixture of frustrating and boring for everyone involved.

I think a few aspects are worth noting. As evinced by Q4, even 'easy' combinatorial questions can be found hard by well-prepared students, especially under pressure. The average mark was < 4, which is unusual, but shows that commentariat calls of "far too easy for an IMO" should, as ever, continue to be ignored, especially in this category.

Although no real knowledge was needed for Q2, technical fluency in case separation and analysis was required to present a convincing solution via the most common method. This effect was considerably magnified in Q5, where the natural language for a successful solution (ie going beyond what I outlined at the start of the discussion in this report), though straightforward enough, would not be familiar to less experienced students. The technical challenges of handling a finite number of thresholds for 'eventually' statements within the various cases are considerable.

It's perhaps not surprising then that this IMO was less rewarding for countries with less extensive preparation programs than the UK is fortunate to be able to offer. In Rio, less than a quarter of the ~ 100 participating countries received no medal among their contestants, and many countries enjoyed comfortably their strongest ever performance, but here, and in Hong Kong 2016 (where the paper was also excellent, in my opinion), more than a third were medal-less.

Overall, while it would be a shame if this effect was heavily magnified, I think it's important to maintain a reasonable spectrum of difficulty within the medium range, and a reasonable balance between entirely-from-first-principles creativity and some technical background. The latter is important since it has greater relevance to the students' futures in mathematics[13] and justifies organising an appropriate level of preparation. And, despite the exciting international collegiate value of the IMO itself, it's during the preparation that both enthusiasm for mathematical endeavour and the real educational value is added.

Background and UK results

For the past few years, a handful of UK students have participated in the IMO three or four times each, and the turnover from one year to the next has been low. 2018 was rather different. Harvey Yau from Carmarthenshire becomes the first British student to take part in five IMOs, following his previous three silver medals and a memorable gold in Rio 2017. However, for the other five students, it's their first IMO. Indeed, Aron and Tom only attended our introductory camp in Oxford last September, and their elevation to this level so rapidly speaks volumes about their hard work over the past months.

[13] and is thematically closer to most research, however relevant one thinks that may be. I think it is 'not irrelevant and probably at least slightly relevant', but a lot less relevant than commentators sometimes like to imply. A conventional undergraduate degree is perfectly sufficient to decouple all flavours and aspects of school-age mathematics (including perceived ability) from what lies beyond, which is, even for past IMO contestants, very much not uniquely restricted to academic mathematical research.

Much of this progress has been collaborative though, and it's worth saying that several other students, including the reserve, Emily Beatty, who obtained a perfect score at the European Girls' Olympiad in Florence in April, were also well-placed to thrive at the IMO, along with some younger students whose enthusiasm bodes well for the future.

Maximising our ranking at the IMO is not the unique focus of the progamme, but in any case, UK came 12th at IMO 2018, with one perfect score, four silver medals, and one honourable mention for a complete correct solution, which was a superb set of results for an inexperienced team, and we are very proud of them. Here are the individual results.

	Q1	Q2	Q3	Q4	Q5	Q6	Σ	
Agnijo Banerjee	7	7	7	7	7	7	42	Perfect Score, Gold Medal
Sam Bealing	7	2	0	2	0	0	11	Honourable Mention
Tom Hillman	7	4	0	7	5	3	26	Silver Medal
Benedict Randall Shaw	7	7	0	7	7	0	28	Silver Medal
Aron Thomas	7	3	0	7	7	1	25	Silver Medal
Harvey Yau	7	7	0	7	7	1	29	Silver Medal

Geoff's report[14] contains many fascinating factoids about the country results, and I highly recommend you look there if you are interested. We will briefly remark here, though, that only one other of the ~ 600 students at IMO 2018 obtained a perfect score, and the last time a UK student achieved this feat was six years before Agnijo was born! 42 is not really the answer to life, the universe, nor everything, but it's nonetheless a wonderful achievement to mark the end of his olympiad 'career', and we wish Agnijo along with Harvey, Sam, Emily and all their Year 13 colleagues from our camps and mentoring programmes every success as they start the next step of their mathematical journey at university in the autumn.

Pedagogical digressions

Our trip began with five days of IMO practice in Budapest, together with the Australian team and their leaders. The mathematical programme comprised a 4.5 hour exam each morning, roughly simulating what they'll face at the IMO itself, and time in the afternoon for discussion, and seminar-style exploration of relevant further topics. All the adults are well aware of the differences between competition problems and both undergraduate study and active research, but some preparation time for the IMO is a good opportunity to focus on skills which will remain useful throughout a mathematical career, and surely beyond too. At most of our other

[14] https://www.imo-register.org.uk/2018-report.pdf

events, 4.5 hour exams form part of a selection process, and so it's harder to use the work produced during such exams to ignite more general self-improvement.

Doing papers simulating the IMO certainly exposes the students to ideas and techniques which might be useful the following week at the competition itself. But more than that, it affords them a chance to reflect on how they and their peers are *writing* and *explaining* proofs. As with many first-year undergraduates, we generally find that IMO contestants are initially not quite formal enough, and certainly not organised enough for their written work on harder problems to be digested easily, but that any attempt to summarise an argument, for example verbally over lunch, will often be far too technical to be digested easily without paper.

Attempting fifteen questions over five mornings gives plenty of opportunity to work on finding an appropriate level of write-up clarity, and the subsequent lunchtimes an equally excellent opportunity to eat deep-fried cheese, and consider how best to outline the most salient ideas in your morning's work. The following question drawn from the IMO 2017 shortlist, appeared on Tuesday's exam, and is an excellent example of both of these.

(IMO Shortlist 2017 N3) *Determine all integers $n \geq 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers*

$$a_i, \quad a_i + a_{i+1}, \quad a_i + a_{i+1} + a_{i+2}, \quad \ldots, \quad a_i + a_{i+1} + \ldots + a_{i+n-1}$$

is divisible by n. (With indices taken modulo n when necessary.)

I don't really want to give away the answer, but suppose we are trying to prove that the statement holds for some value of n, by contradiction. Because of the nature of what we have to prove, it's natural to write the numbers in a circle, and then we are studying the partial sums along various arcs. In particular, given one arc, say $a_i \to a_j$, there's a natural *next arc* starting from a_{j+1}. So it makes sense to consider a sequence of consecutive arcs which, by the contradiction assumption, we may assume all have sum divisible by n. Drawing a diagram at this point will be extremely helpful.

The crucial point is that the sequence of arcs will go round the circle one or more times, but eventually will 'repeat', in the sense that two arcs will have the same starting point. This means that we can refine to a sequence of arcs which cover each number on the circle exactly the same number of times, say k. But each arc has sum divisible by n, so the sum across all arcs is also divisible by n. Since we know the total sum across all numbers is *not* divisible by n, this will cause problems for a particular class of values of n and k.

Turning this into a proof by slightly increasing the formality of each of the sentences above is unlikely to work. At the end of the argument, it's unavoidable that you need to perform a short calculation to get a simple equation, and it's just not possible to form an equation without names for the quantities which will appear on each side! The solution given in the booklet considers $i_0, i_1, i_2 \ldots$ to be the starting points of the 'arcs', so that $a_{i_\ell}, a_{i_\ell+1}, \ldots, a_{i_{\ell+1}-1}$ is a typical arc. Without a diagram, this has less insight, but it does enable you to double count by considering the sum

$$a_{i_\ell} + a_{i_\ell+1} + \ldots + a_{i_{\ell+1}-1}$$

and then summing this sum over all relevant values of ℓ. You might have clarified this by insisting that $i_{\ell+1}$ is *minimal* such that the above sum is a multiple of n. Then, to clarify the range of relevant values of ℓ, you could take L minimal such that that $i_L = i_\ell$ for some $\ell \leq L$, and argue that in fact this means $i_L = i_0$. Both of these steps are rather natural, though in fact neither is absolutely necessary.

If you don't argue properly why i) your arcs eventually repeat, in some way; ii) your arcs cover everything an equal number of times; iii) you have some control over how many times they cover everything, your argument will look unconvincing. You can probably get away with asserting ii) without proof, but i) and iii) will require an argument, which is much better made with notation. You will certainly need to use properties of n and k, and if you have clear notation, you'll hopefully realise right in the middle of this argument if you have the wrong set for n.

However, when explaining at lunch, you will receive glazed expressions if you start defining i_ℓ, especially if you include this minimality condition on $i_{\ell+1}$. It just isn't central to the argument. You are considering consecutive arcs, and covering the whole circle with such arcs equally, going round at most $n-1$ times. The number theory comes in right at the end, involving solutions to an equation or relation for n and k. Getting 7/7 on the IMO is not contingent on being able to provide such a concise summary, but this idea, whether visual or otherwise, is what's useful to take away from this problem beyond the 4.5 hours on Tuesday morning.

Problem composition

Appealing problems for these competitions do not grow on trees, and the many stages of checking for alternative solutions and getting colleagues to assess difficulty all takes time. One has to accept that, even after all this process, lots of potential problems will be put out to grass because either i) they don't have quite the right balance between technical routine and hard-to-spot insight; or ii) they are too similar to something that has been seen before, and there really are onlookers with an encyclopedic knowledge of such prior art!

One of our top 2017 contestants, Joe Benton, seems to have ignited an interest amongst UK IMO participants in composing problems, and we are arriving at a position where we can set our selection tests mostly using problems written by staff and recent students. Indeed, even during the trip, there were several proposals. The main challenge for Aron might well be to keep his latest offering secret from his peers until 2021, when he will have left school. In particular, Sam Bealing has been writing several geometry problems during the year, many of which have stretched me far beyond my geometric comfort zone, and it was nice that he had the opportunity to test one out on the Australians during one of our training camp exams. Since this now can't be used in a future competition, Sam explains its origin, and his solutions, here.

(F3 Q3 - Sam Bealing) *In acute, scalene triangle ABC let H, O, Γ be the orthocentre, circumcentre and circumcircle respectively. Let M be the midpoint of BC and let AH intersect Γ again at D. Ray MH intersects Γ at K. The lines perpendicular to AC, AB through O intersect BC at E, F, respectively. Let X be the intersection of lines OH and MD.*

Prove that if $AKEF$ is cyclic then X also lies on this circle.

The starting point for this problem was thinking about when the circumcentre O lies on the circle with diameter BC. This obviously occurs when $\angle A = 45°$. A circle passing through the circumcentre makes the problem ripe for inversion. E, F play the roles of the inverses of the feet of the altitudes from B, C. A little bit of angle chasing shows these lie on the perpendicular bisector of the sides.

The circle in the problem is actually just the inverse of the circle with diameter AH in $\odot ABC$. By a well-known result[15], this passes through K. A bit of experimenting with Geogebra gave a way of describing the inverse of H without referring to lengths (X).

[15] As we shall see, to handle K, it's useful to consider the point A' on Γ diametrically opposite A.

The difficulty in this problem comes from handling the circle condition and drawing a diagram where the points look cyclic. The problems

$$\begin{aligned} AKEF \text{ cyclic} \quad &\Rightarrow \quad \angle A = 45^\circ \\ \angle A = 45^\circ \quad &\Rightarrow \quad X \text{ on circle} \end{aligned}$$

would both be much easier but putting them together is what makes the given problem difficult.

Solution 1 (Synthetic): We first prove $AKEF$ cyclic implies $\angle A = 45^\circ$, after a lemma.

Lemma: *Let P, Q be the midpoints of AM, EF respectively. Then POQ are collinear.*

Proof of lemma: By angle chasing and observing M, Q are both midpoints we see $\triangle OEF \cup Q \simeq \triangle HBC \cup M$. Using the fact that E, F lie on BC we see $QO \parallel HM$. But by applying a homothety of factor 2 at A, we have $PO \parallel MA'$ where A' is the point diametrically opposite A. So as it is well-known HMA' are collinear the claim follows. $\square$

Claim: if $AKEF$ cyclic then $\angle A = 45^\circ$.

Proof of claim: It's well-known that $\angle AKM = 90^\circ$ and HM passes through A', the point diametrically opposite A. Extend KM to intersect $\odot AEF$ again at X. As AX is a diameter homothoty of factor $\frac{1}{2}$ about A takes line $MA'X$ to POQ by lemma 1. This shows Q is the centre of $\odot AEF$ and hence $\angle EAF = 90^\circ$. Now by angle chasing using $EA = EC, FA = FB$

$$\angle EAF = \angle EAC + \angle FAB - \angle A = \angle B + \angle C - \angle A = 180 - 2\angle A \implies \angle A = 45^\circ. \quad \square$$

We now claim X is the reflection of K in BC. It suffices to prove that if $KD \cap OM = O'$ then O' is the reflection of O in BC. By angle chasing,

$$\angle O'KA' = \angle DKA' = \angle MOA' = \angle O'OA'.$$

So $O'KOA'$ is cyclic. Then, by Power of a Point,

$$O'M \cdot MO = KM \cdot MA' = BM \cdot MC.$$

But as $\angle A = 45^\circ \implies \angle BOC = 90^\circ$ we get $OM = BM$, so $O'M = BM = OM$ as desired.

Circle $AFEK$ has diameter EF, we have $\angle FXE = 90^\circ$, so X also lies on this circle.

Solution 2 (Areals and inversion): The lemma in the previous solution can also be proved via a calculation with areal coordinates.

Alternate proof of lemma: Using the equation for the perpendicular bisector of AC which is $b^2(x - z) + y(a^2 - c^2) = 0$, and similarly for AB we see

$$E = (0, b^2, a^2 - c^2), \quad F = (0, a^2 - b^2, c^2),$$

giving the midpoint of EF as

$$Q = (0,\ a^2S_C + b^2(c^2 - b^2),\ a^2S_B + c^2(b^2 - c^2)).$$

Also $P = (2, 1, 1)$ and $O = (a^2S_A, b^2S_B, c^2S_C)$ so for collinearity it remains to check

$$\begin{vmatrix} 2 & 1 & 1 \\ a^2S_A & b^2S_B & c^2S_C \\ 0 & a^2S_C + b^2(c^2 - b^2) & a^2S_B + c^2(b^2 - c^2) \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ a^2S_A & b^2S_B & c^2S_C \\ 0 & 2b^2S_B - a^2S_A & 2c^2S_C - a^2S_A \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 1 \\ a^2S_A & b^2S_B & c^2S_C \\ -2a^2S_A & -a^2S_A & -a^2S_A \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ a^2S_A & b^2S_B & c^2S_C \\ 0 & 0 & 0 \end{vmatrix} = 0. \quad \square$$

We verify the claim as in the first solution. But to prove X lies on the circle, we use inversion.

We observe $\angle A = 45^\circ \implies \angle BOC = 90^\circ$. Let S, T be the feet of the perpendiculars from B, C to AC, AB respectively then we see $BTOSC$ is cyclic. The relation $\angle A = 45^\circ$ shows that $\triangle ASB$ is isosceles and right-angled which means S lies on the perpendicular bisector of AB. Combining the previous two observations shows that under inversion in $\odot ABC$, $\odot AEF \mapsto \odot AST$.

As $OM = 2R\cos\angle A = \frac{R}{\sqrt{2}}$ we see $M \mapsto O'$, the reflection of O in BC. From this, the line $M'D$ is sent to circle $OM'D$. As OH, DM' are reflections in BC and $OM' \parallel HD$ we see $OM'DH$ is cyclic. Combining these results shows $X \mapsto H$ and as $ASHT$ is cyclic we see that X indeed lies on $\odot AEF$ as desired. $\square$

UK IMO 2018 Diary

Sunday 1st July

We've been spoiled in the past five years, with trips to Colombia, South Africa, Thailand, Hong Kong, and Brazil. The air mile collecting opportunities are reduced in 2018, and so the team is meeting at Luton. The terminal is much changed, but Wizzair is still pink. Unlike in the long haul past, the flight isn't long enough for truly time-consuming problems, though Sam shares a recent American question in which, pleasingly, the bound is achieved via a similar construction to Alice and the Mad Hatter in BMO2 2018 Q2. Our initial impression of Budapest is very positive as, perhaps for the last time for a while, we breeze through the EU entry route.

Monday 2nd July

Mornings this week are occupied by 4.5 hour exams, so after lunch the adults' thoughts turn to marking. The British students have done well, and mostly written very coherently, so Freddie's introduction to IMO-level marking is not a baptism of fire. I'm delighted as six out of six solutions to medium-level geometry is not a situation we have experienced often in the past.

There's time for a short walk round some of the most iconic sites on the Buda side of the Danube, starting with the Liberty statue atop the Gellért citadel, and moving on up the new outdoor escalator to the pink domed castle. Views over Pest provide a scenic backdrop as Agnijo and Aron discuss harmonic ranges with a captive audience of new colleagues from Australia.

As evening falls, the World Cup becomes inescapable. Four years ago, we took 14 year old Harvey to Cape Town for his first IMO, during which he initiated the habit of predicting the outcome of each game by flipping a 10p coin. The coin was wrong every single time. He may be a foot taller now, and actually interested in football, but this seems not to have diluted the magic. His shiny new 200 Forint votes for Mexico, and thus gets off to a good start as Brazil make it $2-0$. Harvey has an offer to study in Cambridge from October where both the Weak and Strong Laws of Large Numbers appear in the first year Probability course. In both cases, the word 'large' is important.

Tuesday 3rd July

Today's exam includes the lovely combinatorial number theory problem discussed earlier, as well as a hard geometry problem, to which Harvey provides a perfect solution. The overall standard of writing is very high, and leaves plenty of time for a quick trip to the Széchenyi baths in the late afternoon. The glorious Habsburg building and its outside courtyard encloses a large number of pools of various appealing temperatures. Tom has found a pine-scented steam room and looks dangerously relaxed, while our Estonian colleague Richard is unimpressed with a mere 80C sauna as, even with the pounding power ballads presumably designed to prevent you falling asleep, apparently his family has an ever warmer one in their block! Either way, after six typically Eastern European meals, my pores are grateful for the exfoliation opportunity.

The non-bathers have been exploring the rest of the Városliget park. Ben has been in charge of the camera, and when Freddie and I later review the flash drive, we are not sure whether, among the many castle and lake panoramas, we are more perturbed by the picture of the sinister hooded 'Statue of Anonymous', or the ultra-close-ups of Aron's face.

For those who are interested, the day concludes with the World Cup. Ben claims to be supporting Colombia, but slightly undermines the strength of this commitment by asking which team is which deep into extra time. Jordan and Eric do their thing, so for the rest of us there is elation, and time for bed.

Wednesday 4th July

To introduce variety into proceedings, today the British and Australian groups have set an exam for each other, so as to experience the joys and travails of marking. This year, several of the students have developed a precocious interest in problem composition, and Sam gets to test two of his hardest offerings so far (see earlier) on a captive audience.

By mid-afternoon, some heads are in hands, with red pens cast aside in temporary bafflement. By early evening, though, all is complete, and Freddie, Andrew and I stage a mock coordination of the results. It's a useful exercise, and I think some of the students have really digested the value of good structure, especially when one needs to quantify the significance of small errors; and the difference between 'do you think the author knows that the the function diverges?' and 'has the author explained to the reader why the function diverges?' Dinner outside in the Jewish quarter is therefore well-earned by all. I sense I am being judged ethically for ordering even a small portion of the iconic Hungarian goose liver, but fortunately they are distracted by a problem about transforming convex polygons, all the more so once it is confirmed that they are allowed to indulge the ultimate stereotype of writing on the table.

Thursday 5th July

After two years living in a non-English-English-speaking country, I'm getting abuse from the British team about my use of phrases such as 'graduating high school', to complement the ongoing drama about the length of the first vowel in 'pedal triangle'. Ben will be representing UK at the International Linguistics Olympiad later in the summer and has robust views on all of this. I guess this is comeback for my crusade against the growing use of 'solve' as a noun[16].

[16] As in 'She's claiming three solves on today's paper'.

The only meaningful linguistic nuance of the morning occurs during discussion with the hotel's catering manager: it seems the Hungarian SI unit of cake is the decigram. 'Savoury cake' is also available, and by the end of marking I feel that 90% of my biomass is cheese pastry, but no chance to burn it off with the onset of a huge thunderstorm. The adjacent conference room is hosting a seminar on 'Critical thinking through board games', and both countries' students have a similar plan for the afternoon. The day ends with Harvey and Australian Ethan taking control of ordering in a Chinese restaurant, whose service is glacial, and whose lazy Susans induce in some of our younger students a level of joy normally reserved for the birth of a first grandchild, or the moment you realise that your ugly partial differential equation separates.

Friday 6th July

The final exam of this camp is traditionally designated *The Mathematical Ashes*, and there is a trophy and a genuine funeral urn full of charred geometry for the winners. Perhaps magnified by the ten years since they last won, the Australians' sense of competitiveness is higher than ours, and following recent events in Cape Town, we're keeping an eye open for sharp practice. Mike Clapper is observed returning from a hardware store with a suspicious yellow package, claiming to have been 'fixing his laptop'...

After lunch, the teams have been sent out to play some escape rooms in the city centre, but the second question is fiddly to mark and we are not quite finished by the time they return, spouting mostly incoherent gibberish about codes and nuclear weapons when we ask whether they had fun. But shortly it's time for Freddie to announce the results, and the winners are... Estonia! Well, at least if you go by average mark, courtesy of our guest Richard. Going by a more conventional total mark, Australia have won back the trophy by a margin $99 - 98$. Aron is already encouraging Tom to start a joint work plan building towards reclamation in 2019.

We eat at the Platán Étterem, who have made the bold choice to honour our booking and open, despite having run out of bread, tomato, and cheese. They claim to be a pizza restaurant. The Australians have an appallingly early flight, so it's a UK-only evening, and a final chance to chat to Freddie, or, for some, to ask him penetrating questions about excircles, before we diverge in various ways after this excellent educational week in Budapest.

Saturday 7th July

We have chosen Budapest because it is reasonably-priced and wonderful, and because it is within a few hours' drive of Cluj-Napoca, across the Great Hungarian Plain. The business end of the IMO is getting started, and many photos of the leaders studying the shortlist are appearing on social media. Tom identifies a few where a question is visible, though fortunately the resolution is low. We wonder whether we could deploy deep learning to reveal the solutions to some of the problems, or at least to discover that they are not a dog?

After an initial misadventure at an unpromising petrol station by the border, our drivers propose stopping for lunch at the first Romanian town, where the move from Forint to Lei seems favourable. Agnijo's dish could be summarised as 'half the ocean on a plate for less than a Central London pint'. Moving onwards, we pass a bizarre village comprising fifty apparently uninhabited faux-oriental four-story houses, which the driver attributes to 'the gypsy mafia'.

Entering the rolling hills of Transylvania, we've run out of geometry and the atmosphere has become slightly restive, even with the occasional excitement of overtaking a horse and cart. Harvey has got good use out of his noise-cancelling headphones. But we make it to Cluj just in time for the second half of England's quarter-final. It seems the organisers have split the teams between five widely-separated hotels, and Sweden is not in ours. Everyone receives a very large number of goodies, including an IMO 2018 polo shirt, an IMO 2018 stress ball, and a slightly baffling IMO 2018 tray. I put a much-needed coffee on it then forget about it forever.

Sunday 8th July

The opening ceremony takes place in a giant sports hall, which we arrive at extremely early. There are reunions with familiar faces amongst the deputy leaders, and time to make new friends. The Australians are making friends via their usual medium of the clip-on toy koala. Aron is keen to attach as many as possible to the straps of my camera. Tumbleweed rolls by.

Eventually the show starts, with seven formal speeches, including both the President and the Vice Prime Minister of Romania. Among the usual pleasantries, and clichés about winners, we learn that 'maths is the Esperanto of the sciences', and a metaphor about dowries, stretched far beyond its tensile limit. The parade of teams is rapid. Harvey is flag-bearer, followed by Sam who gets a hearty high five from the bowtie-wearing bear who is the competition's mascot. Afterwards, Tom seeks out the Swedish team for a brief gloat, and a nice joint photograph.

The afternoon is empty, so there is time to explore downtown Cluj. Both bus travel and the Transylvanian Ethnographic museum are free with an IMO lanyard, so we experience both. The latter starts with some unimpressive farming implements from the 1950s, though it improves as we head upstairs to the traditional peasant costumes, where our sartorial expert Ben is transfixed by some remarkable waistcoats. Everywhere we go, there are bright yellow banners declaring Cluj to be the 'World Capital of Mathematics'. The UK students don't need reminding why they are here, and the main show will start tomorrow, after an early night.

Monday 9th July

The first day of the contest starts early, and the organisers are not happy that a) one of the UK students is bringing a non-official ziploc bag; b) will thus delay departure to the exam site by three minutes while he finds the official version. There is a brief but robust exchange of views in the lobby. Following this, we depart, and the students enter the contest hall 76 minutes before the start of the exam.

A handful of the deputy leaders stay behind to receive a copy of the paper. The Romanians ran this very well, with the right level of invigilation (ie a little bit), and it was really enjoyable to spend a few uninterrupted hours on the problems. As it drifts past the three hour mark, we start to collaborate a bit more. I compare notes with Canadian Calvin on Q3, and our partitioned triangle diagrams are sufficient to convince each other that we have broadly the same solution. Sasha, the US deputy, reprises his previous life as the Ukrainian deputy, and translates this solution into Russian to a captive audience.

There is an hour to write things up outside the exam hall while waiting for the students to emerge. I take the chance to introduce some maximally confusing errors into my solution to Q2. Agnijo is first out, and claims all three problems, which is super news, though the others don't seem so confident about Q2, and haven't spent any time on the final problem. Aron asks me what the silver medal cutoff is likely to be. Sadly, my internal oracle is out of battery. Rather than agonise over the exact details, it seems sensible to find a distraction in town...

...But not a break from time-pressured problem-solving as we've booked another pair of escape rooms, possibly Romania's most famous recent export. I join one of them, and feel distinctly slow as a whir of puzzling activity unfolds around me. Ben demonstrates his ability to remember arbitrarily long sequences of Morse code from one hearing; Tom correctly judges that some exposed wiring is a crucial clue rather than just Eastern European lighting; while anyone doubting Aron's commitment to the physical challenges need only inspect the dust accumulation on his trousers. In the end, we have to ask for one hint as we miss a crucial object taped to the back of a fridge, but still make the top five on the leaderboard, with the other room finishing in a similar blur of excitement shortly afterwards. While top five on the IMO leaderboard may be out of reach this year, hopefully they've saved some of this ingenuity and creativity for tomorrow.

Tuesday 10th July

It's IMO Day Two, and the team are, under heavy coercion in some cases, wearing the lurid green shirts. There are no repeats of yesterday's delays, and so, after an interesting commentary from the bus on the icons of central Cluj from Radu, the irrepressibly upbeat guide for Japan, our highly visible students enter the contest hall 79 minutes before the start of the exam. Later we will hear reports about how this time was spent, and it all sounds very memorable, but perhaps an extra 30 minutes in bed might overall have been preferred?

Andrew shows me a solution to Q6 involving inversion at A, the exotic details of which are still far from settled in my head as we meet the students afterwards. Sam hasn't enjoyed this paper, but we all know that one disappointing day doesn't diminish the value of everything he's achieved and contributed over the past two years. The others seem to have made considerable progress, and Agnijo may be heading towards the UK's highest score for many years.

And so that's that for IMO 2018. Well, for the students. The UK adults gather at the leaders' hotel, which turns out to be only walking distance away. It's good to see Geoff again, as well as compare notes with Ceri, Jeremy, Marina, and Adam, who are all observing aspects of this year's IMO to inform their roles for when the UK hosts the 2019 edition in Bath. With a large and expert team in place, Questions 1 and 2 have already been marked, and it only remains to pick up pdf scans of all the students' work, and discuss some markschemes. It's also great if unsurprising news to hear that Geoff has been re-elected as IMO President, so will have another four years to continue the excellent developments introduced at recent editions.

Meanwhile, the students have been on a tour of the Cluj courtroom, kindly arranged by their guide, Cezara, and her mother, who is the district judge. It turns out that Aron has asked a question during the Day Two exam (in fact two, though neither was 'what's the silver medal cutoff likely to be?'). Harvey has made it through ten IMO papers without he or any previous teammate resorting to this, so appoints himself judge and jury. In Western Romania's trial of the century, following a lengthy and acrimonious summing up, he sentences Aron, who appeals to Tom and Agnijo, unsuccessfully, for clemency. Ben then executes him. I wasn't actually there, but am piecing together from reports of varying coherence given at dinner later.

Later, I join the boys to find out whether France or Belgium will be England's opponents in the [SPOILER ALERT] third-place playoff, before returning to the Day Two pdfs. Q5 has been completed mostly very well, though the technical steps all need careful checking. Evidently this has left considerable time for everyone on Q6, because there are pages and pages and pages to go through. And, like gold in the Yukon, if you look hard enough in the vast expanse, there are marks to be found, or at least it seems so at 3am.

Wednesday 11th July

It's no secret that I like maths and I like arguing, and I really like coordination at the IMO, where the UK leaders meet local markers to agree appropriate and consistent scores for our students. I particularly like coordination in Romania, where the standard of the local markers is absurdly high. The highlight is probably reading through Agnijo's lengthy but careful geometry solution (see earlier), digesting exactly what he's used in each section, and how it might be rewritten to account for all angle orientation cases. Razvan, Pavel and colleagues also agree about the 3am gold nuggets, which will turn out, in a really counterintuitive version of alchemy, to turn two bronzes into two silvers tomorrow.

We are planning to keep the students informed about their marks as the day unfolds. Instead, we end up hearing more from Rosie about their excursion to Alba Iulia, especially when a diversion to a layby in an industrial suburb of Cluj enters its second hour. It sounds mythologically bad[17] but hopefully they might see the funny side afterwards, whether tonight or in years to come.

Excitement builds during the day as it starts to become clear that, in contrast to the fears, Q$\{2,4,5\}$ have not been found easy by many countries, and Q$\{3,6\}$ have been ideally discriminating at the very top end. I thought the question spread was attractive to begin with, but it seems like it's led to an appropriate spread of marks too. It's impossible to keep Agnijo's perfect score quiet once his 7s on the hard questions flash up on the screen, and we receive many congratulations, though of course the hard work has all been his. It seems he will be part of a very select crowd, though the UK is the only top country to squeeze everything in by 6pm, so we will have to wait and see.

[17] Aron's student report doesn't make it sound as excruciating as suggested at the time, but you can read all about it here: `https://www.imo-register.org.uk/2018-report-aron.pdf`

In any case, there is time to take the team out for dinner to celebrate. It seems that even ultra-positive Sam is not yet seeing the funny side of the 1.5 hours in the layby. Agnijo is delighted with his perfect score. He tells us that his next book will include a chapter about olympiads, and receives much advice about how to drop in this fact as subtly (or not) as possible. I make a tongue-in-cheek comment about my attitude to coordination (involving the word 'jaguar') which I suspect will be quoted back to me roughly ∞ times by Aron over the years to come.

Of course, our true intention in finishing coordination today was to be ready to see It (almost) come home. I've now listened to commentary on these England games in German, in Hebrew, in English paraphrased by an Easyjet pilot, in Hungarian, and finally in Romanian. The latter has the most amusing pronounciation of 'Pickford', with at least four syllables. The lobby at the student hotel is full of slightly terrifying men vigorously supporting Croatia. Ben enhances the atmosphere by fetching the Union Jack flagpole provided by the organisers for the opening ceremony. Two hours later, we leave more discreetly.

Thursday 12th July

Since the UK is done with coordination, there's the chance to join the students on their second excursion, this time to Turda salt mine. Another team is rather late turning up, and this reduces the length of today's layby experience to a mere 35 minutes. Ceri is making notes for the 2019 excursions, and "DON'T DO THIS", seems an adequate summary.

Despite the inauspicious start, the salt mine is a genuinely impressive experience. After a short museum explaining the history of the mine, and exhibiting some excavation and extraction equipment, we descend 100m down a spiral staircase. For some, the temptation to lick the walls grows. And what greets us at the floor of the bell-shaped cave? Slightly incongruously, a boating lake, twenty ping-pong tables, and half a funfair. While the students insist on one cycle of the ferris wheel, the mind boggles slightly at how they got it down there, or maybe that was just vertigo and mild nausea? Afterwards, time for a change of axis, as Harvey plays Mao with the Hong Kong team on a playground roundabout, while Ben and Tom increase the angular momentum of the game to a level where even just watching them induces not-so-mild nausea. I retire with Jeremy to watch the teams with less HSE-conscious leaders on the boating lake, very much senza life jackets. Apparently it has far greater buoyancy than the Dead Sea, though a slightly less appealing temperature.

Ceri and I want to diverge back to the leaders' site in time to observe the final jury meeting. The organisers declare that this is literally impossible. Even after we engage Radu's help and phone a taxi, they stick to this assessment. We produce a 50 Lei note (roughly £10), and the plan is hastily reclassified as entirely possible. We make it in time and the meeting is brief. No marks are challenged. One set of potential medal boundaries fits the criteria almost exactly. The UK's interest is the silver threshold, which ends up at 25, on the right side for Aron and Tom. American James Lin is confirmed as Agnijo's only companion on 42/42, and US leader Po-Shen gladly receives much congratulation. After the success of our joint camp, it's great to see Australia and UK in 11th and 12th places overall, which has come as a small but pleasant surprise to all of us! We reunite the students on the hill near the Australians' hotel. The teams empty the restaurant of its unique dessert, a bizarre lavender mousse, and Agnijo gets a scenic background for his press release photo as the sun sets over Cluj.

Friday 13th July

To coincide with hosting the IMO in 2019, the UK is launching the *Diamond Mathematical Challenge*, a programme making it possible for other countries to run their own instalment of our ever-popular Team Maths Challenge using pre-existing material. Rosie runs a demonstration for interested leaders at their site. The UK team get to play the role of teachers. Ben seems to enjoy wielding his red pen, while some of the participating leaders demonstrate that though speed of arithmetic might decline in middle age, a sense of competitive edge need not.

The closing ceremony is uncharacteristically early, though our departure is characteristically earlier still. The French team have brought a football, and the UK team have been given $\sim 10^4$ small stickers to distribute, advertising IMO 2019. With almost two hours to kill, it's fair to say that not all these stickers are distributed in an entirely serious fashion. While we wait for the ceremony to start, Sam and I have a fascinating conversation with one of the Syrian contestants in the row behind about the challenge of following the World Cup qualification from Damascus. Eventually, it kicks off, with seven formal speeches, followed by a brisk presentation of medals.

There is some confusion about how 98 behaves modulo 12, so courtesy of the country alphabet, Aron receives the first silver with only one Turkish boy for company. Subsequent medallists are much less lonely on the stage, many with giant flags as usual, though Harvey's small Welsh dragon captures the eye just as much. Agnijo's reward for his recent perfection is to go up as one of the final pair, and receives a triple-cheek kiss and lengthy embrace from the Vice Prime Minister. After some traditional dancing, the whole UK delegation returns to stage so that Geoff can formally receive the IMO flag from the Romanians. There follows an amusing promotional video for IMO 2019, and then an eardrum-testing laser display ending with a burst of golden ticker tape from the cannon which had been ominously parked next to our row throughout.

A closing dinner is held at the leaders' hotel. Most teams change into casualwear, and it's good to see how much use some of the British boys have made of particular items of UKMT clothing during this trip... Adults and 'children' are vigorously separated; Marina and I generate confusion. The food is identical, though the adult room is serenaded by a quartet of heavily bearded tenors, who would have benefitted from an absence of microphones. Audience participation is encouraged, and the Italian leaders offer a Neapolitan duet which is truly memorable. In the children's room, the serious amps are out, and while Sam heads for the dance floor, Agnijo and Ben declare an interest to move somewhere where it is less audible, for example back to Budapest. Tom has declined to play violin in National Youth Orchestra in favour of the IMO, and we're not sure what they'll make of his apparent new interest in Romanian Drum'n'Bass.

Saturday 14th July

Some of us stay up all night, so Aron doesn't get a final rendition of *Tosca* from Ben's phone alarm. We kill some time perusing *Lemmas in Euclidean Geometry* before moving to Bananagrams. By 3am, the legitimacy of some of Sam and Tom's proposed words is becoming questionable, but it's time to leave. Half the IMO is trying to check in at Cluj airport at 4am, and they seem ill-prepared. I recall a similar experience at Santa Marta in 2013, but Geoff assures me that was far worse. However, the coffee was certainly better and cheaper in Colombia. Nonetheless, several of the team enjoy a triple espresso here, so I'm sure their parents will enjoy the comedown from that later this afternoon.

My own flight to Tel Aviv left five minutes later and I was awake for neither take-off nor landing.

Conclusion

Training a UK team and taking them to the IMO requires a huge amount of effort from a large number of people. Thanks are particularly due to:

- All the staff at our camps in Oxford, Hungary, Cambridge and Tonbridge. Nothing can quite compare to getting so many teenagers with the same interest in the same places, and helping them improve together, and we couldn't do this without all the volunteers happy to give up their time to support this. Thanks also to Bev at the UKMT office, who arranges everything for the IMO trip and our other events during the year so tirelessly!
- The organisers of IMO 2018, who put on an excellent week, of which the high standard of coordination was a particular personal highlight.
- The UK's guides, Cezara and Alex, who went to great lengths to ensure our team was informed and entertained. We wish them all the best for the rest of their time at school.
- Our Australian colleagues, Angelo, Andrew, Mike and Jo, with whom we ran perhaps our most successful ever joint training camp, which gave our students a great chance to share social and academic experiences during the final stages of IMO preparation. We are looking forward to hosting on home soil in 2019!
- The British observers, Rosie, Ceri, Jeremy, Adam, James, and Marina, for all their help with marking, and with supervising and entertaining the students, and their careful observation to inform preparation for what will surely be an excellent IMO in Bath in 2019.
- Freddie, for his help marking and energising the students' academic preparation, especially at the pre-IMO camp; and Geoff, who continues to do a superb job, both behind the scenes and in public, of his roles as IMO President, and also UK leader, and we remain excited to see what further progress and innovation the next four years will bring.
- Finally, of course, our UK team comprising Agnijo, Sam, Tom, Ben, Aron, and Harvey. They had a mature but energetic attitude towards preparation and the competition itself, and were good company throughout the trip and excellent ambassadors for the UK, and for mathematics.