Romanian Masters of Mathematics 2011

UK leader's report

Bucharest, Romania

Introduction

The UK sent a team to the Fourth Edition of the Romanian Master of Mathematics, which took place between 23rd and 28th February 2011. This is a hard international competition for countries with strong mathematical traditions; the UK have been pleased to accept invitations to all four competitions so far.

The team consisted of:

Andrew Carlotti	Sir Roger Manwood's School, Kent	Year 11
Benjamin Elliott	Godalming College, Surrey	Year 13
Richard Freeland	Winchester College, Hampshire	Year 13
Edward Godfrey	Thomas Hardye School, Dorset	Year 13
Adam Goucher	Netherthorpe School, Derbyshire	Year 12
Jordan Millar	Regent House School, County Down	Year 13.

I led the team. However, over the last ten years or so, other countries have started to expect that a UK delegation will be staffed by at least one person with a sense of humour. For that reason the deputy was Mr James Gazet, of Eton College.

Sever Moldoveanu, of Tudor Vianu high school, prepared some interesting statistics based on last year's Romanian Masters competition. From them I extracted the following table comparing the performances of the 32 students who competed both in the Romanian Masters and at the IMO in 2010:

RMM 2010 \ IMO 2010	Gold	Silver	Bronze	Hon. ment.	No award
Gold	4	1	0	0	0
Silver	3	2	2	0	0
Bronze	0	4	6	1	0
Hon. ment.	0	0	0	0	0
No award	1	4	0	3	1

Notice how close this matrix is to being lower-triangular: this provides evidence that this is a truly hard competition.

Questions

The questions in the competition were as follows:

1. Prove that there exist two functions $f, g\colon \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

(Poland) Andrzej Komisarski & Marcin Kuczma

2. Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

   (1) for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;

   (2) the degree of f is less than n.

(Hungary) Géza Kós

3. A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

(Russia) Vasily Mokin & Fedor Ivlev

4. Given a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, we write $\Omega(n)$ for the total number $\sum_{i=1}^{s} \alpha_i$ of prime factors of n, counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ (so, for example, $\lambda(12) = \lambda(2^2 \cdot 3^1) = (-1)^{2+1} = -1$).

   Prove the following two claims:

   i) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = +1$;

   ii) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = -1$.

(Romania) Dan Schwarz

5. For every $n \geq 3$, determine all the configurations of n distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points X_i, X_j there exists a permutation σ of the integers $\{1, \ldots, n\}$, such that $\mathrm{d}(X_i, X_k) = \mathrm{d}(X_j, X_{\sigma(k)})$ for all $1 \leq k \leq n$.

   (We write $\mathrm{d}(X, Y)$ to denote the distance between points X and Y.)

(United Kingdom) Luke Betts

6. The cells of a square 2011×2011 array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).

   Determine the largest positive integer M such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least M.[1]

(Romania) Dan Schwarz

Note that Luke Betts, the author of Question 5, was himself a competitor at the Romanian Masters only last year. He is to be congratulated for his efforts in the capacity of poacher-turned-gamekeeper; I hope it inspires others to do likewise.

[1] Cells with coordinates (x, y) and (x', y') are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.

Results

Here is the highly creditable performance of the UK team:

name	1	2	3	4	5	6	total	
Andrew Carlotti	6	1	1	7	0	4	19	Silver medal
Ben Elliott	6	2	0	7	2	0	17	Bronze medal
Richard Freeland	7	1	0	7	0	3	18	Silver medal
Edward Godfrey	0	3	0	7	0	0	10	Honourable mention
Adam Goucher	7	1	0	7	7	3	25	Gold medal
Jordan Millar	5	1	0	7	7	3	23	Silver medal
total, all six	31	9	1	42	16	13	112	
total, top three	18	3	1	21	14	10	67	

The medal cutoffs were 25 for gold, 18 for silver, and 13 for bronze. Eight gold, twelve silver and twenty-four bronze medals were awarded.

The team competition is determined by the sum of the top three scores of the team. The top teams were as follows:

United States of America	72	1
Hungary	67	2=
United Kingdom	67	2=
Russia	66	4
China	65	5
Serbia	62	6
Italy	58	7
Ukraine	55	8
Romania	47	9
Bulgaria	46	10
Poland	42	11
Brazil	37	12
Peru	30	13

Had it been a competition of total team scores, we would again have been second, this time just behind Russia on 113 and just ahead of China with 110. Both Hungary with 102 and the USA with 99 would lag behind considerably.

Had it been a competition to get consistent scores, things would have been very different. Peru (with scores from 10 to 7) and Romania (with scores from 16 to 13) would have been joint first, while Serbia (with scores from 33 to 2) would have languished at the bottom. Congratulations are due to the Serbian contestant Teodor von Burg, who achieved that top competition score of 33.

The Russian and Brazilian teams both had a girl as their strongest student.

The highest score achieved on Problem 3 by any contestant was only 2; this means our apparently weak performance was, in fact, the common lot.

Leader's Diary

Wednesday 23rd February

Only a moderately early start in Leicester seems to be required, so I amble over to the railway station to get to Luton Airport. Since I want, as is understandable, a return ticket to Luton Airport, I naturally ask for a "return ticket to Luton Airport". The girl at the ticket desk takes the initiative by giving me instead a return ticket to Luton Airport Parkway: this differs from the correct ticket by having the cryptic letters "PW" in the small print, and by not permitting me to travel to the airport.

Rather later, when I discover what has happened, I buy an extra ticket to complete the journey, and resolve to employ my new-found powers to subject the staff of East Midlands Trains to international humiliation in this Diary.

I meet five of the students, and James, at the airport. Unfortunately the convenient meeting point depicted on the airport map on the internet does not exist in real life. Everyone understands anyway.

The journey from Luton to Bucharest Baneasa is uneventful. Jordan even manages to get some sleep: since he lives in Northern Ireland, he has already been travelling for many hours.

It turns out that Romania is blanketed in snow. Fresh snow will fall on almost every day we are there.

We are picked up promptly and shown into a minibus. This already contains the Polish team: they are new to the competition this year, but I know their leader Michał Pilipczuk already from IMO 2010 in Kazakhstan.

From there we are rapidly brought into the city. Our students are staying in student accommodation at the campus of the local University of Economics; James and I are staying in a hotel a couple of minutes' walk away. We meet up with our sixth student, Ben Elliott, who has made alternative arrangements for travel from South Africa where he had been taking a holiday.

After checking in, we meet on campus for a dinner of fried fish, rice, beetroot and bread. There we spend some time introducing James to friends from previous competitions.

Dan Schwarz, the mastermind behind the competition, arrives. After expressing alarm at the Frenchness of Mr Gazet's surname, he explains that this year he has been only half of the problem selection committee: the other half was Ilya Bogdanov, from Russia.

The UK staff are given a memory stick containing competition details, and retire to consider the draft paper it contains. The questions are pretty, and also hard. I am very pleased to note that Dan has recognised my heritage by putting the flag of the Bailiwick of Guernsey on the list of participants.

Thursday 24th February

After breakfast, the leaders meet at a respectable time, to walk over to Tudor Vianu, the elite Bucharest school where many olympiad mathematicians are educated, and where the competition is based.

We are there for a jury meeting, to talk about the paper. Many changes are discussed, but very few are made. The only substantial change is that it is decided to make Problem 1 somewhat easier: it began with the words "Determine whether there exist", and we take pity on our students by replacing it with "Prove that there exist".

After this, James and I make trivial efforts to improve the quality of the English of the draft. Given the high standards of written English of the problem selection committee, and the modest standards displayed by us, we have few suggestions and the process is quick.

Much of our time is spent on Problem 6, where a large variety of different approaches are explored. How do we describe the array of cells, and what does it mean to be a neighbour? In the end we decide to go for both extremes: a motivating explanation, and a gritty formal characterisation using modular arithmetic.

This brings us to a late lunch, and then the opening ceremony.

Upon being reunited, our students tell me that they have had a fruitless morning. Their guides took them to the Parliament building, but found they had to show photographic ID to get in. But James and I had locked their passports in a safe, and so they were ejected from the premises by armed guards.

The opening ceremony is fairly brief. This is impressive, since there are four competitions going on in parallel: besides the well-established mathematics competition, there are smaller numbers of students attempting to compete at chemistry, physics and informatics. Several boring old people give short speeches, made lively by internecine strife between the various disciplines, and then two exciting young people compère an introduction to the teams.

Then it is back to the jury meeting. I can sit in peace while everyone else translates our English version into their preferred languages. Dan wonders if we should produce a language in some native British language other than English. However, happily for us, Richard is not a Welsh speaker and Jordan has also declined to request versions in Ulster Scots or Irish.

The evening passes readily. The students attempt some IMO 2009 hard combinatorics shortlist problems together, under the stern supervision of Mr Gazet. They demolish C8 in good time. The two of us then retire to give the students an early night, and to sample a small quantity of Romanian beer.

Friday 25th February

We breakfast and get to Tudor Vianu for a 9:30am competition start. Dan continues his windup by stopping Jordan outside the exam room and checking that his English is up to the task, apologising profusely for the lack of papers in other languages native to Northern Ireland.

In the first half-hour students may submit written questions. Our team bombard us with them, with students commonly asking, "What does $f \circ g$ mean?", "Is 0 deemed to be divisible by every integer n?", and "What is meant by the common inner tangents to a pair of circles?". Zoli, from Hungary, assures me that modern teaching pedagogy states that it is a bad thing when students stop asking questions.

After some thumb-twiddling I go for lunch, then meet the students after the paper. There is general satisfaction. I then retire to collect their scripts and evaluate them.

A lengthy evening is spent in my shared room with James, in the style of Eric and Ernie, passing scripts between us.

There are no embarrassments at all: five of the students have done Question 1, and Ed has had a good bash at Question 2. A couple of other attempts at Question 2 are seriously flawed, and there are no solutions at all to Question 3.

Saturday 26th February

Today is the second paper. So, after breakfast with the Brazilians, we head over to take questions. There are fewer today.

After that, while the students are sitting the paper, we are going through coordination: the collaborative marking effort where we agree scores with the locals.

Problem 3 is first. The coordinators are evidently bored, and so we investigate our zeroes in detail. They are eventually persuaded to award us a mark for some speculative diagram decoration by Carlotti: otherwise they would not be awarding many marks at all.

After an hour or so we do Problem 1. We clash over Carlotti's script immediately: they wish it to receive zero, and I feel that seven might be more appropriate. In the end, the coordinators drive a hard bargain, but they do it in a friendly and evidently fair fashion. We accept our marks (two 7s, two 6s, including the Carlotti script, and a 5) gracefully. It turns out we are the top country at Problem 1.

Then we do Problem 2. The two coordinators ask us what we want. I announce my dream marks. One of the coordinators looks at his notes, and wordlessly makes a series of gestures that we interpret as Romanian Sign Language for "Gosh, that is low enough that we cannot argue with you about it". We sign the form, shake hands and leave.

We are in a decent position. Four students have eight marks, which is a strong start. The two others are well off the starting blocks.

This gives us a few hours to have a leisurely lunch with the students. It is clear that they have sunk their teeth into the problems firmly and repeatedly. Sometimes this has produced solutions, and sometimes it has produced mess, but it is clear that their gung-ho approach is commendable.

After this we go off to collect the scripts. These are very encouraging indeed. Everybody has done Problem 4 and we have two solutions for Problem 5 (by Adam and Jordan). There is a convoluted script by Ben which makes some progress on Problem 5, but which is seriously incomplete and fraught with difficulties, and an outline of an argument by Andrew which contains many excellent ideas for Problem 6.

I take a break from reading the scripts to dine with an old friend in downtown Bucharest.

Sunday 27th February

We wake early. James goes to see the students; I trudge through fresh snow to Tudor Vianu to prepare for coordination. I have just finished a pass over the Problem 4 scripts when I am ushered in to coordination for them.

They ask what we want. I say, "six sevens", trying not to use the same voice as when I say, "the ace of trumps". They agree immediately: it is entirely to our students' credit that they have all written their scripts in such a way that the coordinators can find nothing to object to.

Then comes Problem 6. Three of the students have not made any serious attempt at a proof, but have engaged hard enough to guess an optimal configuration. This is well rewarded. Carlotti gets an extra mark for daring to try a proof.

An hour or so later we finish with Problem 5. We quickly get Adam and Jordan their big marks, and then fight for a few for Ben's partial attempt.

I wander around, feeling fairly satisfied. I wonder if we may have done enough to beat one or two strong European countries: Roberto from Italy had earlier suggested we might beat them, for example. Then Dan comes to shake my hand, congratulating me on my team's strong performance. He says we have beaten Russia. I find myself wondering why the Russians have performed so poorly. Eventually (for I am slow) I realise what he means. Russia have not been weak; we have been strong.

I spend some time walking around being happy, and then phone Geoff Smith, whom I picture sitting eagerly next to the phone like an expectant father.

It is time to see the team. We take lunch together. Five of the students reveal their embarrassment: they had believed that "Adam P. Goucher" was a fictitious human being designed by the UKMT Politbüro to spur them on. How easily we have persuaded them otherwise!

We go next to the closing ceremony. The team are happy to receive their prizes. I decide to spruce up my United Kingdom credentials by receiving our "second place" certificate with a large Union Flag.

Razvan Gelca, the American leader, receives the plate which is the trophy for the winner. Thus the USA will add their name to the list of winners, underneath the UK, China and Russia.

A journalist seeks an interview with me. Besides being charmingly well-presented, I am impressed by her complete failure to ask any daft questions. In the UK, one might expect such an interview to be coloured by an unjustified preconception that mathematics is an intrinsically silly activity.

Dan gives a speech full of kind words to the students, leaders and deputies, which rather more bluntly signals his retirement from the organisation of the competition. We must record our permanent gratitude to him.

In the evening, there is the final party. We can at last relax and licentiously discuss differing education systems.

Monday 28th February

James and I rise at 0530 (or, as it's known in the UK, 0330) to prepare to leave. The friendly Ukrainians accompany us as far as our airport.

The students report that there were parties rather late, and that they felt obliged to bid farewell to the Peruvian team when they left in the middle of the night. Several of them do indeed look the worse for wear by the time we successfully repatriate them.

Conclusion

Many thanks and congratulations are due to many people.

These include (and are certainly not limited to) the following:

- Sever Moldoveanu for his excellent organisational work at Tudor Vianu;
- Dan Schwarz and Ilya Bogdanov for compiling two very beautiful papers;
- the students of Tudor Vianu for entertaining our team in a warm and friendly manner;
- the staff at the UKMT office for efficiently making all the necessary logistical arrangements at our end;
- Joseph Myers and Geoff Smith for being pivotal in drumming up support from back home;
- the students' families for their constant behind-the-scenes support (and a kindly offer of a lift home from Luton);
- the students' schools for arranging to release them during termtime so that they could compete;
- James Gazet for his helpful, hard-working attitude and companionship; and (of course)
- Andrew, Ben, Richard, Edward, Adam and Jordan, for working so hard and for being such easy company.

James Cranch (`jdc41@cam.ac.uk`), 1st March 2011.