INTERNATIONAL MATHEMATICAL COMPETITION
in MERSCH (LUXEMBOURG)

First day : THURSDAY JULY 10th 1980

Time : 4 hours

(1) Find all functions f from Q to Q satisfying the following two conditions:

(i) $f(1) = 2$

(ii) $f(xy) = f(x)f(y) - f(x+y) + 1$ for all x,y in Q.

[Q is the set of all rational numbers.]

(2) Let A,B,C be three collinear points with B between A and C. On the same side of AC are drawn the three semi-circles on AB, BC and AC as diameters. The common tangent at B to the first two semi-circles meets the third at E. Let U and V be the points of contact of the other common tangent of the first two semi-circles.

Calculate the ratio

$$\frac{\text{area of triangle EUV}}{\text{area of triangle EAC}}$$

as a function of $r_1 = \frac{1}{2}AB$ and $r_2 = \frac{1}{2}BC$.

(3) Let p be a prime number and n a positive integer. Prove that the following statements (i) and (ii) are equivalent:

(i) None of the binomial coefficients $\binom{n}{k}$ for $k = 0,1,\ldots,n$ is divisible by p.

(ii) n can be represented in the form $n = p^s q - 1$ where s and q are integers, $s \geq 0$, $0 < q < p$.

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(4) Two circles touch (externally or internally) at the point P. A line touching one of the circles at A cuts the other circle at B and C. Prove that the line PA is one of the bisectors of the angle BPC.

(5) Ten gamblers started playing each with the same amount of money. Each in turn threw five dice. At each stage the gambler who had thrown paid to each of his nine opponents $\frac{1}{n}$ times the amount which that opponent owned at that moment, where n is the total shown by the dice. They threw and paid one after the other. At the tenth throwing the dice showed a total of 12, and after payment it turned out that every gambler had the same sum as he had had at the beginning. Determine if possible the totals shown by the dice at each of the other throwings.

(6) Determine all pairs (x,y) of integers satisfying the equation

$$x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1).$$