RESEARCH STATEMENT

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1. Background

I study symplectic manifolds and their symmetries. These are smooth spaces with a skew pairing on their tangent directions, like the pairing of coordinates and momenta in classical dynamics. My principal interest is in complex projective varieties: spaces arising as the zero sets of homogeneous polynomials over the complex numbers, which inherit a symplectic pairing from the ambient projective space. The symplectic structure becomes relevant when considering deformations: varying the polynomials will usually give a different projective variety, but the symplectic structure is unchanged and yields some very subtle deformation invariants (for example Gromov-Witten invariants which count rational curves).

Algebraic geometers have introduced and studied *moduli spaces* which parametrise all possible deformations of a variety. Even in the case of a complex curve this moduli space is very rich and interesting. Teichmüller theory reduces the study of its topology to understanding the *mapping class group* of the curve. This is a discrete group of smooth symmetries (diffeomorphisms) considered up to isotopy. The natural generalisation of this to higher dimensions uses symplectic geometry: the analogue of the mapping class group is the *symplectic mapping class group* (and more generally the *symplectomorphism group*). However, the theories of moduli spaces and of symplectic mapping class groups of higher dimensional varieties are not as well-developed as the case of complex curves, so a precise correspondence has not been established. It is this relationship which motivates many of the questions I study.

2. Previous work

My previous work studies the topology of the symplectomorphism group for particular complex surfaces (4-dimensional spaces). The first computations of this kind were by Gromov [6] who showed that the symplectomorphism group of the quadric surface retracts onto its subgroup of rotations. Later Abreu [1], Seidel [9], Lalonde-Pinsonnault [8] and others calculated the topology of the symplectomorphism group for some very simple rational surfaces. I proved the following theorem, which extended the scope of calculations to some highly non-trivial (though still rational) examples:

Theorem 1 ([5]). *Let $\mathbb{D}_n$ denote the n-point blow-up of $\mathbb{CP}^2$ considered as a monotone symplectic manifold and write $Symp_0$ for the group of symplectomorphisms acting trivially on homology.*

- *$Symp_0(\mathbb{D}_2)$ and $Symp_0(\mathbb{D}_3)$ are both homotopy equivalent to the 2-torus.*
- *$Symp_0(\mathbb{D}_4)$ is contractible.*
- *$Symp_0(\mathbb{D}_5)$ is homotopy equivalent to the pure braid group on 5 strands on the sphere.*
- *If A_n denotes the 2-dimensional Milnor fibre*

$$\{x^2 + y^2 + z^n = 1\} \subset \mathbb{C}^3$$

considered with the pullback symplectic form from $\mathbb{C}^3$ then the group of compactly supported symplectomorphisms $Symp_c(A_n)$ is weakly homotopy equivalent to its group of components. If $n \geq 4$ this component group is isomorphic to the braid group on n strands.

The Milnor fibre calculation verifies a special case of a conjectural relationshipbetween the topology of the symplectomorphism group and the topology of the "space of stability conditions on the derived Fukaya category" as computed for this example by Professor Richard Thomas in 2006. This exciting relationship is suggested by considerations in string theory, specifically the mysterious "string duality" known as mirror symmetry. The derived Fukaya category is an algebraic object built out of a class of special submanifolds in the symplectic manifold called *Lagrangian submanifolds*, which play a central role in open string theory.

Another part of my doctoral work was a classification of these Lagrangian submanifolds in certain 4-dimensional symplectic manifolds. This was first achieved for Lagrangian spheres in the quadric surface by Hind [7]. My contribution was to extend this classification to some more complicated complex surfaces:

Theorem 2 ([4]). *Any pair of homologous Lagrangian spheres in the 2-, 3- or 4-point blow-up of $\mathbb{CP}^2$ (with its monotone symplectic form) are isotopic through Lagrangian embeddings.*

3. Future plans

The methods used in my thesis are not powerful enough to pin down the relationship between moduli spaces of varieties and symplectomorphism groups in full generality. New analytical ideas are needed for progress to be made. Here are some approaches I will explore.

- Given a symplectic manifold there is a contractible space of associated metrics which contains the moduli space of the underlying variety as a subset (of Kähler metrics). The *Nijenhuis energy* is a functional on this space which measures how far one is from this Kähler locus. The downward gradient flow of this functional might in good cases exhibit a retraction onto the Kähler locus, which is closely related to the moduli space of the variety. There are many related unanswered questions about this functional: can we find manifolds for which the only critical points are Kähler? Are there ever non-Kähler manifolds with non-zero infimum of the functional? It is likely that the Euler-Lagrange equations for the functional form part of an elliptic system after gauge-fixing (work in progress), which would indicate good analytic properties. This approach is inspired by the beautiful papers [2], [3].
- It would be very exciting to test the conjectural relationship betwen symplectomorphism groups and spaces of stability conditions in examples other than the Milnor fibres. There are K3 surfaces (quartic hypersurfaces in projective 3-space) for which the space of stability conditions is known. Using Seidel's homological mirror theorem for K3s and Bridgeland's calculation of the space of stability conditions it should be possible to prove that the symplectic mapping class group of a K3 is not finitely generated (in contrast to theorem 1).
- The examples where we can compute the symplectic mapping class group are probably atypically simple. This promotes a longer term question: Is there a geometrically defined class of manifolds for which the homotopy type of the symplectic mapping class group is not algorithmically computable? Is there a class for which it is computable? For example finite-type Stein manifolds? Our understanding of these issues is in its infancy.

References

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