

1 Elliptic Functions

Definition 1 (Elliptic function). A non-constant meromorphic function f is said to be elliptic if there exists τ with $\text{Im}(\tau) \neq 0$ such that $f(z+1) = f(z) = f(z+\tau)$ for all $z \in \mathbb{C}$. Set $\Lambda = \{a+b\tau : a, b \in \mathbb{Z}\}$. Then we say f is Λ -periodic.

For $\xi \in \mathbb{C}$, define the fundamental region \mathcal{P} to be the parallelogram with vertices $\xi, \xi+1, \xi+\tau, \xi+1+\tau$. WLOG I take $\xi = 0$.

Theorem 1 (Constraints on elliptic functions). Let f be an elliptic function with fundamental region \mathcal{P} . Since the algebraic zeros a_1, a_2, \dots, a_k and algebraic poles b_1, b_2, \dots, b_ℓ of f are isolated we may assume none of them lie on the boundary of \mathcal{P} .

1. f is holomorphic implies that f is constant,
2. the sum of the residues at the poles is zero,
3. the number of poles equals the number of zeros (this number is called the degree of f , $\deg f$),
4. $\sum a_i - \sum b_j \in \Lambda$
5. an elliptic function takes on the value $c \in \mathbb{C}$ $\deg f$ times on \mathcal{P} .

Proof. 1. This is trivial from Liouville's theorem.

2. The residue theorem tells us that the sum of the residues equals the integral of the elliptic function around the boundary of \mathcal{P} , but since f is periodic the contributions from opposite faces of the boundary cancel to give zero.
3. The argument principle relates the difference $k - \ell$ to the integral

$$\frac{1}{2\pi i} \int_{\partial \mathcal{P}} \frac{f'(z)}{f(z)} dz$$

which vanishes by the same reasoning as above.

4. Consider the integral $\frac{1}{2\pi i} \int_{\partial \mathcal{P}} z \frac{f'(z)}{f(z)} dz$. The residue theorem tells us that this is just $\sum a_i - \sum b_j$. Integrating directly over pairs of opposite sides, we get:

$$-\frac{\tau}{2\pi i} \int_0^1 \frac{f'(z)}{f(z)} dz = -\tau n(F \circ \gamma_1, 0)$$

and similarly for the slanting sides γ_2 , where $\gamma_1(t) = t, \gamma_2(t) = t\tau$. Each of these paths is closed since f is doubly-periodic and thus has integer winding number around 0.

5. $f(z) - c$ is an elliptic function for any $c \in \mathbb{C}$ with $\deg f$ poles. Thus it has $\deg f$ solutions in \mathcal{P} (counted with multiplicities). □

We now introduce theta functions, used in the construction of elliptic functions.

Definition 2 (Theta functions). Define the theta function

$$\theta(z, \tau) = \sum_{n=-\infty}^{\infty} \exp\left(\frac{1}{2}n^2\tau + nz\right)$$

This is a well-defined, even holomorphic function, as may be checked by applying the Weierstrass M -test locally on \mathbb{C} . Usually we omit the τ from the notation.

Theta functions have period 1, but are not elliptic. Instead they satisfy

$$\theta(z + \tau) = \exp(-\tau/2 - z)\theta(z)$$

The argument principle tells us that $\theta(z)$ has exactly one simple zero in \mathcal{P} . Indeed we can use evenness of θ to show that this zero is precisely at $(1 + \tau)/2$.

Theorem 2 (Existence of elliptic functions). Given points $a_1, \dots, a_k \in \mathcal{P} \setminus \partial\mathcal{P}$, $b_1, \dots, b_k \in \mathcal{P} \setminus \partial\mathcal{P}$ satisfying $\sum a_i - \sum b_j \in \Lambda$ and such that $a_i \neq b_j$ for all i and j , there exists an elliptic function f , unique up to multiplication by a constant, with poles at each b_i and zeros at each a_i .

Proof. Begin by constructing such a function. Set $g(z) = \theta z - 1/2 - \tau/2$ and define

$$\tilde{f}(z) = \frac{g(z - a_1)g(z - a_2) \cdots g(z - a_k)}{g(z - b_1)g(z - b_2) \cdots g(z - b_k)}$$

This has period 1 and $\tilde{f}(z + \tau) = \exp(m\tau)\tilde{f}(z)$ for some integer m . So $f(z) = \exp(-mz)\tilde{f}(z)$ is the required elliptic function. If there were two such functions then their ratio would be elliptic and holomorphic on \mathcal{P} , therefore constant. \square

2 Weierstrass's \wp -function

Definition 3. Weierstrass's \wp -function is defined to be the unique elliptic function of degree 2 with a double pole at 0 such that the Laurent series has the form $z^{-2} + az^{-1} + bz + \dots$.

To see that this function is uniquely determined, suppose there were two. Their difference would have poles only at points of Λ and the z^{-2} terms would cancel, leaving at most a single pole. But then it would be either holomorphic (hence constant, and the condition on the constant term ensures it vanishes) or of degree 1, which would mean it had one zero, which cannot occur since the sum of residues must vanish. Arguing in a similar fashion about $\wp(z) - \wp(-z)$ tells us that \wp is even. So much for its properties, how do we know such a function exists?

Theorem 3. *The Weierstrass \wp -function can be expressed as*

$$\wp(z) = A - \frac{d}{dz} \left(\frac{\theta'(z - \frac{1}{2} - \frac{\tau}{2})}{\theta(z - \frac{1}{2} - \frac{\tau}{2})} \right)$$

Proof. This is a meromorphic function and a simple calculation shows it to be τ -periodic. $f(z) = \theta(z - 1/2 - \tau/2)$ has a simple zero at 0, so if $f(z) = zg(z)$ with $g(0) \neq 0$ then $-\frac{d}{dz} \left(\frac{f'}{f} \right) = z^{-2} - (g'/g)'$ so A can be chosen to cancel the constant term. \square

The derivative \wp' is a degree 3 elliptic function with triple poles at all lattice points and simple zeros at $1/2$, $\tau/2$ and $(1 + \tau)/2$. Thus the valency $v_\wp(z)$ is 2 if $z - 1/2$, $z - \tau/2$ or $z - (1 + \tau)/2$ are lattice points.

Now $\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} (2k-1)E_{2k}z^{2k-2}$ for some coefficients E_{2k} , so a good deal of symbol shuffling tells us that $\wp'(z)^2 = 4\wp(z)^3 - 60E_4\wp(z) - 140E_6$. This equation is of crucial importance, as it links together the theories of elliptic functions, elliptic curves and elliptic integrals. For it implies:

$$\wp(z) - \wp(z_0) = \int_{z_0}^z \frac{dz}{\sqrt{4z^3 - 60E_4z - 140E_6}}$$

and shows that any complex affine cubic in Weierstrass form $y^2 = x^3 + ax + b$ can be parametrised by elliptic functions.

Finally, we note that the ring of elliptic functions on a given lattice form a field, and that

Theorem 4. *If f is an even elliptic function then $f(z) = R(\wp(z))$ for some rational map R and if f is a general elliptic function then $f(z) = \wp'(z)R_1(\wp(z)) + R_2(\wp(z))$ for rational functions R_1 and R_2 .*

Proof. We prove only the second part, assuming the first. Any elliptic function $f(z)$ can be written as a sum $f_0(z) + f_1(z)$ with f_0 even and f_1 odd. The even part can be written as a rational function of \wp . f_1/\wp' is even, and so can be written as a rational function of \wp . The result follows. \square