Symplectic geometry and the Nijenhuis energy

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Symplectic manifolds

A symplectic manifold is a pair (X, ω) where X is a $2n$-dimensional manifold and ω is a closed, nondegenerate 2-form.

Example

Phase space of classical dynamics:

$$X = \{(x, y, z, p_x, p_y, p_z)\}, \ \omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z$$

A Hamiltonian function on phase space gives a time-evolution $\phi_t : X \to X$. *The crucial observation is that*

$$\phi_t^* \omega = \omega$$

Such a map is called a symplectomorphism. Classical mechanics acts through symplectomorphisms.

Projective varieties

More relevant example...

Example

Complex projective varieties:

$$\{x \in \mathbb{CP}^n : p_1(x) = p_2(x) = \cdots = p_k(x) = 0\}$$

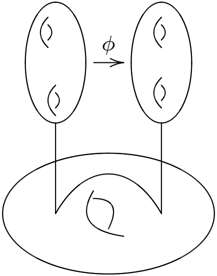
homogeneous complex polynomials

$\mathbb{CP}^n$ has a Kähler form which induces a Kähler form ω on any complex projective variety. Kähler forms are symplectic.

Remembering this symplectic structure gives us a new perspective on old problems in algebraic geometry.

Moduli problems

Since Riemann, algebraic geometers have studied families of varieties obtained by varying the polynomials. If every member of this family is smooth then the topology doesn't change between fibres. In fact, more is true. Given a path in the space S parametrising the family one obtains a symplectomorphism ϕ relating the fibres at its endpoints.



In particular this gives a map

$$\pi_1(S) \to \pi_0(\mathrm{Symp}(X))$$

where Symp(X) is the infinite-dimensional group of all symplectomorphisms. Thus symplectic geometry allows us to probe the topology of families of varieties.

Aim

To understand Symp(X) for complex projective varieties X.

I focus on the case of projective surfaces (4-dimensional symplectic manifolds) as they are the only case where we can say anything at all.

Classification

One can identify four types of complex surface by the *Kodaira dimension* (analogous to the classification of curves by genus).

Kod. dim.	Example
$-\infty$	Fano: $\mathbb{CP}^2$, $S^2 \times S^2$, $\mathbb{D}_n$ $(n < 8)$
0	$K3$, complex tori, ...
1	Elliptically fibred, ...
2	General type

One can hope for a moduli space with finitely many components parametrising all surfaces in a given class (with fixed Chern numbers). We can use symplectic topology to probe the homotopy groups of the moduli space as outlined in the previous slide.

Examples

My thesis contains many examples. Here is arguably the most beautiful and complicated.

Example

$\mathbb{D}_5$ is the 5-point blow-up of $\mathbb{CP}^2$. By varying the five points one obtains a universal family. The parameter space S has fundamental group the group $PBr_5(S^2)$ of pure 5-braids on the sphere. To see this, note that five points lie on a unique conic and therefore determine a configuration of five points on the sphere.

Theorem (Evans)

$$\pi_0(Symp(\mathbb{D}_5)) \cong PBr_5(S^2)$$
$$\pi_i(Symp(\mathbb{D}_5)) = 0 \quad (i > 5)$$

This is proved using Gromov's theory of pseudoholomorphic curves.

The techniques used to prove this theorem work well in the Kodaira dimension $-\infty$ class of surfaces where rational curves are abundant and regular. If we want to say anything about the other classes we need completely new technology. This proposal outlines one approach which has never been tried. The questions it raises are as diverse and interesting as the potential applications to the topology of symplectomorphism groups.

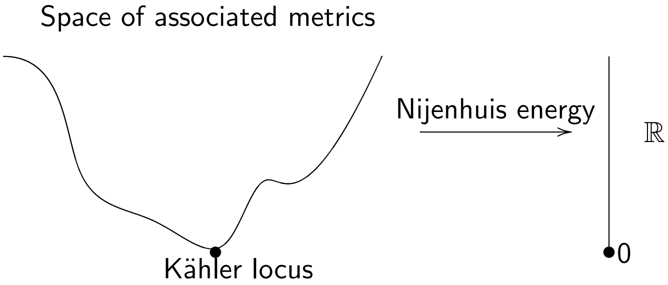
Setup

Definition

To a symplectic manifold one can associate a space of associated metrics, $\mathcal{J}$*.*

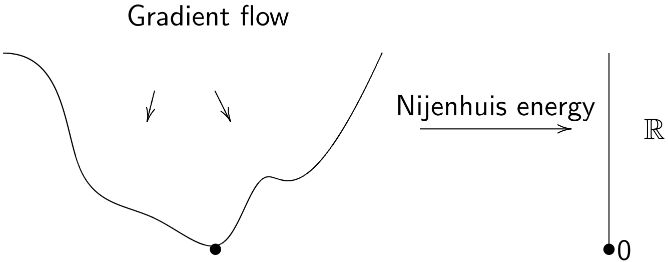
- *These generalise Kähler metrics: the space of Kähler metrics is contained in* $\mathcal{J}$*.*
- *They form a contractible, infinite-dimensional space on which the group* $Symp(X)$ *acts.*

Nijenhuis energy



The Nijenhuis energy is a functional on the space $\mathcal{J}$ of associated metrics which measures the deviation of an associated metric from being Kähler.

Ricci-Hermitian flow



We might hope that the gradient flow of this functional (the *Ricci-Hermitian flow*) retracts $\mathcal{J}$ onto the subspace of Kähler metrics. This flow is

$$\frac{dg_{ij}}{dt} = R_{ij} - {\omega_i}^a {\omega_j}^b R_{ab}$$

and short-time existence and uniqueness of solutions was established by Lê and Wang (2001).

- This retraction would imply a strong relationship between the topology of the symplectomorphism group and that of the space of Kähler metrics. However, geometric flows rarely behave so well.
- The flow might get hung up on non-Kähler critical points (Ricci-Hermitian metrics). There are no known examples of non-Kähler Ricci-Hermitian metrics in dimension four.
- Work of Abreu-Granja-Kitchloo tells us that for $S^2 \times S^2$ the inclusion of the Kähler locus into $\mathcal{J}$ is a homotopy equivalence. The results of my thesis suggest this might also be true for $\mathbb{D}_3$, $\mathbb{D}_4$ and $\mathbb{D}_5$.

This leads to the tentative

Conjecture

On a Fano surface there are no non-Kähler Ricci-Hermitian metrics.

Singularities

Often, solutions to geometric flows become singular. It is likely that this happens for the Ricci-Hermitian flow. To understand singularity formation it makes sense to look at examples where we know something goes badly wrong. For example, symplectic manifolds which admit no Kähler metric.

Example (Kodaira-Thurston)

There is a non-trivial T^2*-bundle over* T^2 *which admits a symplectic form but no Kähler metric. Explicit solutions to the Ricci-Hermitian flow tend asymptotically to a singular metric which is degenerate in two directions: this can be seen as a Gromov-Hausdorff collapse to a torus.*

Even the simplest questions are unanswered: for a non-Kähler symplectic manifold, what is the infimum of the Nijenhuis energy over the space $\mathcal{J}$? Is it realised by a metric?

Example

For the Kodaira-Thurston manifold the infimum is zero.

There are many other examples of non-Kähler symplectic manifolds obtained as twisted fibre sums of Lefschetz fibrations. It would be intriguing to use Donaldson's approximately holomorphic techniques to prove:

Conjecture

The infimum is zero for all twisted fibre sums.

Examples

The analysis of the flow is very hard. Examples with symmetry should guide us. The following example was motivated by the case of toric surfaces.

Example

Consider $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ *with linear coordinates* $(\underline{x}, \underline{y})$ *and the symplectic form* $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. *Restrict attention to associated metrics of the block-diagonal form*

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

Lemma

This class of metrics is preserved by the Ricci-Hermitian flow.