BMO 2007–2008 Round 2
Problem 3—Generalisation and Bounds

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1 Introduction

Problem 3 (by Paul Jefferys) is:

> 3. Adrian has drawn a circle in the xy-plane whose radius is a positive integer at most 2008. The origin lies somewhere inside the circle. You are allowed to ask him questions of the form "Is the point (x, y) inside your circle?" After each question he will answer truthfully "yes" or "no". Show that it is always possible to deduce the radius of the circle after at most sixty questions. [*Note: Any point which lies exactly on the circle may be considered to lie inside the circle.*]

This problem generalises naturally to other dimensions; the following generalisation is considered here:

> Positive integers d and n are given. Adrian has chosen a positive integer $r \leq n$ and a closed ball B of radius r in d-dimensional Euclidean space, containing the origin. You are allowed to ask him questions of the form "Is the point $\mathbf{x}$ in B?" After each question he will answer truthfully "yes" or "no". Let $Q(d, n)$ be the least integer such that it is always possible to deduce r after at most $Q(d, n)$ questions. Determine $Q(d, n)$.

The original problem then asks for a proof that $Q(2, 2008) \leq 60$. We will determine the asymptotic behaviour of $Q(d, n)$, showing that

$$|Q(d, n) - ((d + 3)/2) \log_2 n|$$

is bounded by a function of d.

2 Upper bounds

Taking $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_d$ as a standard basis of mutually orthogonal unit vectors for $\mathbb{R}^d$, we write the general point in $\mathbb{R}^d$ as $\mathbf{x} = (x_1, x_2, \ldots, x_d) = \sum_{i=1}^{d} x_i \mathbf{e}_i$. We often need to refer to the intersections of the surface of the ball B with the coordinate axes: we write x_i^+ for the greatest value of x_i such that $x_i \mathbf{e_i} \in B$, and x_i^- for the least such x_i; we write x_i^c for the coordinates of the centre of B. When in the xy-plane, we similarly use notation such as x^+.

We make repeated use of the following lemma in a series of planes to bound the values x_i^c, x_i^+ and x_i^-, together with questions forming a binary search along coordinate axes in order to bound values x_i^+ and x_i^- sufficiently well for the required applications of this lemma.

Lemma 2.1 *Let real numbers $x_{\max} \geq u_x > 0$ and $y_{\max} \geq u_y > 0$ be given. Consider the closed balls B in $\mathbb{R}^2$ such that $-x_{\max} \leq x^- \leq -x_{\max} + u_x$, $x_{\max} - u_x \leq x^+ \leq x_{\max}$ and $y_{\max} - u_y \leq y^+ \leq y_{\max}$, and such that at least one of the following holds: either $y_{\max} - u_y \geq x_{\max}$, or $(0, -x_{\max}) \notin B$. All such balls have $-u_x/2 \leq x^c \leq u_x/2$, and there exist y_1, y_2 with $y_1 \leq y_2 \leq y_1 + u_y + 3u_x$ such that all such balls have $y_1 \leq y^- \leq y_2$.*

Proof Clearly $x^c = (x^- + x^+)/2$, from which the bounds on x^c are obvious; it remains to prove the bounds on y^-.

First consider the case where $y_{\max} - u_y \geq x_{\max}$ (so $y^+ \geq x^+ > 0$ for all balls B satisfying the given conditions). By the symmetry of the conditions of the lemma, the set of possible values of y^- is the same for balls with $x^c \geq 0$ as it is for balls with $x^c \leq 0$, so suppose without loss of generality that $x^c \geq 0$.

Now the centre of B is the intersection of the line $x = x^c$ with the perpendicular bisector of the line between $(x^+, 0)$ and $(0, y^+)$; this is $(x^c, (x^+/y^+)x^c + y^+/2 - (x^+)^2/2y^+)$. Since $y^- = 2y^c - y^+$, we deduce that $y^- = 2(x^+/y^+)x^c - (x^+)^2/y^+$.

We need to prove that $\max y^- - \min y^- \leq u_y + 3u_x$. We have

$$
\begin{aligned}
\max y^- - \min y^- &\leq \max 2(x^+/y^+)x^c - \min 2(x^+/y^+)x^c \\
&\quad + \max(x^+)^2/y^+ - \min(x^+)^2/y^+ \\
&\leq u_x + \frac{x_{\max}^2}{y_{\max} - u_y} - \frac{(x_{\max} - u_x)^2}{y_{\max}} \\
&= u_x + x_{\max}^2 \left(\frac{1}{y_{\max} - u_y} - \frac{1}{y_{\max}} \right) + \frac{x_{\max}^2 - (x_{\max} - u_x)^2}{y_{\max}} \\
&= u_x + \frac{x_{\max}^2 u_y}{y_{\max}(y_{\max} - u_y)} + \frac{2x_{\max}u_x - u_x^2}{y_{\max}} \\
&\leq u_y + 3u_x
\end{aligned}
$$

so the result is proved in the first case.

Now consider the case where $y_{\max} - u_y < x_{\max}$, so $(0, -x_{\max}) \notin B$ (whence $y^- > -x_{\max}$). Because $|x^c| \le u_x/2$, we have $y^+ - y^- > 2r - u_x$, so $y^- < y^+ + u_x - 2r$. Now we have $y^+ \le y_{\max} < u_y + x_{\max}$, and $r \ge x_{\max} - u_x$, so $y^- < u_y + x_{\max} + u_x - 2(x_{\max} - u_x) = u_y + 3u_x - x_{\max}$, so the result is proved in the second case as well. □

Theorem 2.2 *For all positive integers d and n, we have*

$$Q(d,n) \le ((d+3)/2)\log_2 n + (d/2)(\log_2 d) + 7d + 3.$$

Proof If $d = 1$, within $4 + 2\lfloor \log_2 n \rfloor$ questions we may determine both x^- and x^+ to within ranges of $2n/2^{1+\lfloor \log_2 2n \rfloor} < 1$, and so determine the integer value of r, so suppose now that $d \ge 2$. We may also clearly suppose that $n > 1$.

We define $s = \max\{r, 2\}$ and $u = 2\sqrt{s - 1/4}/\sqrt{d-1}$. For a succession of appropriately translated coordinate systems (such that the origin of the translated coordinates is still known to be in B), we will determine x_i^- and x_i^+ to within intervals of length at most u, for $1 \le i \le d-1$; in those coordinates, we will have $-u/2 \le x_i^c \le u/2$, for $1 \le i \le d-1$. (After we translate the x_{i+1}-coordinates, the bounds on x_i^+ and x_i^- will no longer apply in the new coordinate system, but the bounds on x_i^c will still apply.) Thus the location of the centre of B projected orthogonally onto the subspace generated by $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$, $\mathbf{e}_{d-1}$ will have been determined to be within a distance of $\sqrt{s-1/4}$ from the origin. From this, it follows that if $r > 1$ then $0 \le r - (x_d^+ - x_d^-)/2 \le 1/2$, since $(r-1/2)^2 + (r - 1/4) = r^2$. Thus it will suffice to determine each of x_d^+ and x_d^- to within an interval of length less than $1/2$: we will then have found a lower bound on $(x_d^+ - x_d^-)/2$ that is greater than $r-1$ (or possibly equal to $r-1$ if $r = 1$) and less than or equal to r.

We determine x_i^+ (for $1 \le i \le d-1$) and x_1^- to within intervals of length at most u by a binary search among intervals whose length increases with distance from the origin. Let $a_0 = 0$ and, for $i \ge 1$, define $a_i = a_{i-1} + 2\sqrt{7/4}/\sqrt{d-1}$ if $a_{i-1} < 4$ and $a_i = a_{i-1} + 2\sqrt{\lceil a_{i-1}/2 \rceil - 1/4}/\sqrt{d-1}$ if $a_{i-1} \ge 4$. If $x_i^+ \ge a_j$ then $r \ge \lceil a_j/2 \rceil$, so confining x_i^+ to one of these intervals does suffice to confine it to an interval of length at most u.

If $a_k > 2n$, we can confine x_i^+ to one of the intervals within $\lceil \log_2 k \rceil$ questions, and we wish to bound k. If $a_i = 2b^2 + 1/4 \ge 4$ (so $b \ge \sqrt{15/8}$), we have $2(b+1)^2 - 2b^2 = 4b + 2 < 6b$, so $a_j > 2(b+1)^2 + 1/4$ for $j \ge i + \lceil 3\sqrt{d-1} \rceil$. Since also $a_j > 4$ for $j = \lceil 3\sqrt{d-1} \rceil$, we conclude that $k \le \lceil 3\sqrt{d-1} \rceil \lceil \sqrt{n} \rceil$. So at most $\left(1 + \log_2 3\sqrt{d-1}\right) + (1 + \log_2 \sqrt{n}) \le 4 + (1/2)(\log_2 d + \log_2 n)$ questions are needed to find x_i^+ to within an interval of length at most u.

The algorithm now proceeds as follows. First we determine bounds on x_1^+ and x_1^- by the above method. One of the intervals found may be smaller than the other; if so, we extend the smaller interval away from the origin to the length of the longer interval, so we have a symmetric configuration again, and call the longer interval length u_1. We then translate the x_1-coordinates so that $\max x_1^+ + \min x_1^- = 0$, and we know that the origin is still inside B in the translated coordinates.

Now, for $1 \leq i \leq d-2$, we suppose inductively that the origin is still inside B in the current coordinates and that we know that $-\max x_i^+ \leq x_i^- \leq -(\max x_i^+) + u_i \leq 0 \leq \max x_i^+ - u_i \leq x_i^+ \leq \max x_i^+$. Ask whether $(\max x_i^+)\mathbf{e}_{i+1}$ is in B; if not, negate the x_{i+1}-coordinates. We now apply a binary search, as above, to determine x_{i+1}^+ to within an interval of length at most u (with lower bound at least 0, and at least $\max x_i^+$ if the answer to the previous question was "yes"). Applying Lemma 2.1, we also know x_{i+1}^- to within an interval of length at most $4u$ (upper bound at most 0), so with two more questions we know x_{i+1}^- to within an interval of length at most u. Let the larger length of the two intervals be u_{i+1}, and extend the shorter interval away from the origin and translate the origin as was done for x_1 above.

Now we have bounded x_i^c to within the required intervals, for $1 \leq i \leq d-1$. This has been done within $d(4 + (1/2)(\log_2 d + \log_2 n))$ questions for the binary searches along the axes, $d-2$ questions to decide whether to negate coordinates, and $2(d-2)$ questions to reduce the intervals for x_{i+1}^- values, a total of $7d - 6 + (d/2)(\log_2 d + \log_2 n)$ questions. It remains to bound x_d^+ and x_d^- to within intervals of length less than $1/2$.

To do so, we ask whether $(\max x_{d-1}^+)\mathbf{e}_d$ is in B; if not, negate the x_d-coordinates. Then do a binary search for x_d^+, this time to within an interval of length less than $1/2$ rather than at most u. Then by Lemma 2.1 we know x_d^- to within an interval of length at most $(1/2) + 3u$, and apply a binary search to reduce this to an interval of length less than $1/2$. The number of questions involved in these final steps is at most 1 for the decision on negating coordinates, $3 + \log_2 n$ to find the interval for x_d^+, and $5 + (1/2)\log_2 n$ to find the interval for x_d^-, a total of $9 + (3/2)\log_2 n$, giving the total number of questions for the whole algorithm as specified above. $\square$

To achieve a slight improvement on the bound for the original problem, note that in the above the choice of taking $1/2$ of the possible error in the estimate for r from the first $d-1$ coordinates, and $1/2$ from the last coordinate, was arbitrary. In this particular case, it proves better to take $u = 2\sqrt{s/2 - 1/16}$, so error of $1/4$ comes from the first coordinate and up to $3/4$ from the second. With this choice, $a_{127} > 4016$; seven questions are made to approximate each of x_1^+ and x_1^-, one to decide whether to negate x_2, twelve to approximate x_2^+ and nine to approximate x_2^-, so yielding $Q(2, 2008) \leq 36$.

3 Lower bounds

Suppose that, after some number of questions, the radius r can be deduced from the answers. In addition to determining the radius, the answers will have determined a set of possible positions for the centre of the ball B; this set will be a measurable subset of the closed ball of radius r centred at the origin.

Suppose the answers are consistent with some nonempty set S of closed balls as possibilities for B. Then they are also consistent with any closed ball B' of positive integer radius less than or equal to n that is contained in the union of the balls in S and contains the intersection of those balls (any "yes" answers must have related to points in the intersection, and any "no" answers must have related to points outside the union; the origin must be contained in the intersection). Thus all closed balls of positive integer radius less than or equal to n, contained in the union of the balls in S and containing the intersection of the balls in S, must have radius equal to r. We will show that this leads to upper bounds on the volume of the set of possible positions for the centre of B, and so to lower bounds on the number of questions required.

Lemma 3.1 *Suppose $d \geq 2$. If the answers to the questions have determined the radius to be r, then the diameter of the set of possible positions for the centre of B is at most $2\sqrt{2r-1}$. (The diameter of a nonempty set of points is the supremum of distances between pairs of points in the set.)*

Proof The result is trivial for $r = 1$, so suppose $r > 1$. First note that any two balls consistent with the answers to the questions must intersect, if only at the origin. Suppose there are two whose centres are distance $2s$ apart, where $\sqrt{2r-1} \leq s \leq r$. The surfaces of these balls intersect in a sphere of radius $\sqrt{r^2 - s^2} \leq r - 1$ (in a $(d-1)$-dimensional subspace), and it is easy to see there is a closed ball of radius $r - 1$ (in fact two such balls, unless $s = \sqrt{2r-1}$) whose surface contains that sphere, and which contains the intersection of the two balls and is contained in their union. $\square$

Lemma 3.2 *Suppose a convex set S in $d \geq 2$ dimensions has volume V and diameter R. Then it contains an open ball of radius at least $V/2R^{d-1}$.*

Proof This proof is based on that in [1] for a similar result for a set of circumradius R. (In high dimensions, a ball of radius R has lower volume than the cube of side R used in this proof to contain a set of diameter R, and so that result would give slightly better bounds.)

Let r be the largest radius of an open ball centred at the centre of mass of S and contained in S. This ball must touch the boundary of S at some point P, and because S is convex there is a $(d-1)$-dimensional plane passing through any point on the boundary of S, such that all of S not lying in that plane lies on the same side of it. Without loss of generality, let P be the origin, let the plane be $x_1 = 0$ and let S all lie in $x_1 \geq 0$; then the centre of the ball is $r\mathbf{e}_1$, and this is the centre of mass of S. Because the diameter of S is R, we may also without loss of generality suppose S to lie in $0 \leq x_i \leq R$ for all $1 \leq i \leq d$.

If $r < V/2R^{d-1}$, we have $V > 2rR^{d-1}$, so some part of S of positive measure lies in $x_1 > 2r$. Let the volume of the part that lies in $x_1 \leq 2r$ be V_0, and move some measurable subset of the part lying in $x_1 > 2r$, of volume $2rR^{d-1} - V_0$, into $x_1 \leq 2r$, possibly losing convexity in the process (but keeping the bounds on all the coordinates). Since a positive volume has moved to lower x_1-coordinates, the centre of mass of the resulting set has x_1-coordinate less than r, but since the new set contains all of $0 \leq x_1 \leq 2r$, $0 \leq x_i \leq R$ for $2 \leq i \leq d$ (possibly minus some set of measure zero), plus some positive volume with $x_1 > 2r$, the centre of mass has $x_1 > r$, a contradiction. Thus $r \geq V/2R^{d-1}$. $\square$

Lemma 3.3 *Suppose $d \geq 2$, and suppose an integer $r \geq 12$ is given. Let S be a nonempty set of points in d-dimensional space with diameter at most $2\sqrt{2r-1}$. Suppose there exists a point $\mathbf{p}$ (not necessarily in S) such that, for every unit vector $\mathbf{v}$, there is a point $\mathbf{x}$ in S with $(\mathbf{x}-\mathbf{p})\cdot\mathbf{v} \geq 2+12/r$. Then the closed ball B of radius $r-2$ centred at $\mathbf{p}$ contains the intersection of the closed balls of radius r whose centres are the points of S, and is contained in the union of those balls.*

Proof Without loss of generality let $\mathbf{p}$ be the origin. The conditions then clearly imply that the origin lies inside the convex hull of S, which in turn implies that $\|\mathbf{x}\| \leq 2\sqrt{2r-1}$ for all $\mathbf{x}$ in S.

To see that B contains the intersection of the given balls, write a general point outside of B as $s\mathbf{v}$, where $s > r-2$ and $\mathbf{v}$ is a unit vector. There is some point $\mathbf{x}$ in S with $\mathbf{x} \cdot -\mathbf{v} \geq 2+12/r$, and the point $s\mathbf{v}$ must lie outside the ball with radius r centred at $\mathbf{x}$.

To see that B is contained in the union of the given balls, write a general point of B as $s\mathbf{v}$, where $0 \le s \le r-2$ and $\mathbf{v}$ is a unit vector. If $s \le r - 2\sqrt{2r-1}$, then it is contained in all the balls, so suppose $r-2\sqrt{2r-1} < s \le r-2$. Now consider the ball centred at the point $\mathbf{x}$ of S with $\mathbf{x}\cdot\mathbf{v} \ge 2+12/r$. Write $\mathbf{x} = a\mathbf{v} + b\mathbf{w}$, with $\mathbf{w}$ being a unit vector orthogonal to $\mathbf{v}$ and $b \ge 0$; we have $2+12/r \le a \le 2\sqrt{2r-1}$ and $0 \le b \le 2\sqrt{2r-1}$. If $a - \sqrt{r^2-b^2} \le s \le a + \sqrt{r^2-b^2}$ then $s\mathbf{v}$ will be in the ball under consideration. We have $\sqrt{r^2-b^2} \ge \sqrt{r^2-8r+4}$, so it will suffice if $2\sqrt{2r-1} - \sqrt{r^2-8r+4} \le r - 2\sqrt{2r-1}$ and $r-2 \le 2+12/r+\sqrt{r^2-8r+4}$; both of these hold for $r \ge 12$. $\square$

Theorem 3.4 *For all positive integers d and n, we have*

$$Q(d,n) \ge ((d+3)/2)\log_2 n - ((d+2)/2)\log_2 d - (d+11)/2.$$

Proof If $d=1$, then consider the set of centres determined by questions that have determined the radius to be $r>1$. This set has diameter at most 2, otherwise it would be consistent with a ball of radius $r-1$. Thus the questions must distinguish at least $n(n+1)/2$ possibilities, requiring at least $2\log_2 n - 1$ questions.

Now suppose $d>1$. The set of possible positions for a centre of a closed ball of radius r is a closed ball of radius r centred at the origin, and a trivial lower bound for the volume of this ball is that of a cube of side $2r/\sqrt{d}$, inscribed in this ball; that is, $(2r)^d d^{-d/2}$.

Consider a subset of this set that is the set of possible positions for the centre once the radius has been determined to be r after some number of questions, and suppose $r \ge 12$. By Lemma 3.1 this subset (and so its convex hull) has diameter at most $2\sqrt{2r-1}$. Suppose this subset has volume V. Its convex hull then contains an open ball of radius at least $V/2\left(2\sqrt{2r-1}\right)^{d-1}$, by Lemma 3.2. If this radius is greater than $2+12/r$, then the convex hull satisfies the conditions of Lemma 3.3; but the existence of $\mathbf{x}$ in the convex hull with $(\mathbf{x}-\mathbf{p})\cdot\mathbf{v} \ge 2+12/r$ implies the existence of such $\mathbf{x}$ in the original set, so the original set satisfies the conditions of Lemma 3.3 and the set is consistent with a ball of radius $r-1$, a contradiction.

Thus our subset has volume at most

$$2(2+12/r)\left(2\sqrt{2r-1}\right)^{d-1} < 8\left(2\sqrt{2r-1}\right)^{d-1},$$

so the questions must distinguish at least

$$(2r)^d d^{-d/2}/8\left(2\sqrt{2r-1}\right)^{d-1} \ge 2^{-(d+3)/2} r^{(d+1)/2} d^{-d/2}$$

possibilities for balls of radius r, if $r \geq 12$. So if $n \geq 12$ then the total number of possibilities is at least

$$2^{-(d+3)/2} d^{-d/2} \int_{11}^{n} r^{(d+1)/2} dr,$$

that is,

$$2^{-(d+3)/2} d^{-d/2} (2/(d+3)) \left(n^{(d+3)/2} - 11^{(d+3)/2}\right).$$

Now, $(11/12)^{5/2} < 7/8$, and $d+3 < 4d$, so there are at least

$$2^{-(d+11)/2} d^{-(d+2)/2} n^{(d+3)/2}$$

possibilities. Distinguishing them will require at least

$$((d+3)/2) \log_2 n - ((d+2)/2) \log_2 d - (d+11)/2$$

questions. Finally, if $n < 12$ the given bound is less than 1, and is negative if $n = 1$, so the result holds for $n < 12$ as well. $\square$

Applying this to the original problem, the closed ball of radius r has area πr^2, and the subsets have area at most $4(2 + 12/r)\sqrt{2r-1} < 8(1 + 6/r)\sqrt{2r}$, so at least

$$\frac{\pi}{8\sqrt{2}} \int_{11}^{2008} \frac{r^{3/2}}{1 + 6/r} dr > \frac{\pi}{8\sqrt{2}} \int_{60}^{2008} \frac{r^{3/2}}{1.1} dr > 2^{24}$$

possibilities must be distinguished. Thus $Q(2, 2008) \geq 25$.

4 Further refinements

Above, $Q(d,n)$ was found to be $((d+3)/2)\log_2 n$ to within $(d/2)(\log_2 d) + O(d)$. The following outline arguments suggest that the $(d/2)(\log_2 d)$ terms can be eliminated, leaving an uncertainty $O(d)$, which would be harder to eliminate.

In high dimensions, a cube inscribed in a sphere gives a much better approximation to that sphere's volume than a cube circumscribing that sphere. The $(d/2)(\log_2 d)$ terms arise from approximations by inscribed cubes (which present little opportunity for further improvements), and can be eliminated by avoiding approximations of other spheres by circumscribing cubes. In the lower bound, use of a sphere rather than a cube in Lemma 3.2 (i.e., using the lemma from [1] directly) would yield the required improvement; further slight improvements in the $O(d)$ term would be obtained by using any better bounds available on the volume of a body with given diameter. In the upper bound, searching each dimension in turn to locate the centre means the centre is effectively being located within a cube rather than a sphere; the search in each dimension should be adjusted so that it uses slightly more questions when finding a bigger lower bound on the radius (where subsequent dimensions will be more constrained) than when finding a smaller lower bound on the radius. It may be shown that if s successive binary searches are to be used to locate something in s dimensions from N possibilities (by determining its value on each axis in turn through a series of questions dividing the range of possible values on that axis at some point on that axis), this can be done in less than $2s - 1 + \log_2 N$ questions, so the binary search in each dimension can indeed be arranged to take account of the number of possibilities remaining for subsequent searches after each possible result of the search in that dimension, with $O(d)$ overhead.

The remaining $O(d)$ difference between upper and lower bounds would then reflect factors including issues with logarithms not being integers and associated overheads in binary search, and differences in volume between the largest $(d-1)$-dimensional set with given diameter (used in the lower bound) and the cube with the same diameter (used in the upper bound).

References

[1] G. Rote and R. F. Tichy, *Quasi-Monte-Carlo Methods and the Dispersion of Point Sequences*, Mathl. Comput. Modelling **23** (1996), no. 8–9, 9–23.