THE XXI INTERNATIONAL MATHEMATICAL OLYMPIAD
LONDON 1979

Problems selected for the consideration of the Jury.

An asterisk * before a problem indicates that it is a departure from tradition which the Jury may like to consider.

BI In the Euclidean plane every regular polygon having an even number of sides can be dissected with lozenges.
(A lozenge is a quadrilateral whose 4 sides are all of equal length).

*B4 From a bag containing 5 pairs of socks, each pair a different colour. a random sample of 4 single socks is drawn. Any complete pairs in the sample are discarded or replaced by a new draw from the bag. The process continues until the bag is empty or there are 4 socks of different colours held outside the bag. What is the probability of the last alternative.

BGI Find all polynomials $f(x)$ with real coefficients, for which

$$f(x)f(2x^2)=f(2x^3+x).$$

BG3 A pentagonal prism $A_1A_2\ldots A_5B_1B_2\ldots B_5$ is given. The edges, the diagonals of the lateral walls and the internal diagonals of the prism are all coloured in either red or green, in such a way that no triangle, whose vertices are vertices of the prism, has its three edges of the same colour. Prove that all edges of the bases are of the same colour.

CS2 Let $n \geqslant 2$ be an integer. Find the maximal cardinality of a set M of pairs (j,k) of integers, $1 \leqslant j < k \leqslant n$, with the following property :

$$\text{if } (j,k) \in M \text{, then } (k,m) \notin M \text{ for any } m.$$

*CS4 Find the real values of p for which the equation in x

$$\sqrt{2p+1-x^2} + \sqrt{3x+p+4} = \sqrt{x^2+9x+3p+q}$$

has exactly 2 real distinct roots. ($\sqrt{t}$ means the positive square root of t).

FRGI Given that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319} = \frac{p}{q}$,
where p and q are natural numbers having no common factor, prove that p is divisible by 1979.

FRG3 For all rational x satisfying $0 \leqslant x < 1$, f is defined by

$$f(x) = \begin{cases} \frac{1}{4}f(2x), & \text{for } 0 \leqslant x < \frac{1}{2} \\ \frac{3}{4} + \frac{1}{4}f(2x-1), & \text{for } \frac{1}{2} \leqslant x < 1. \end{cases}$$

Given that $x = 0{\cdot}b_1b_2b_3\ldots$ is the binary representation of x, find $f(x)$.

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FRG4 S and F are two opposite vertices of a regular 8-gon. A counter starts at S and each second is moved to one of the two neighbouring vertices of the 8-gon. The direction is determined by the toss of a coin. The process ends when the counter reaches F. a_n is the number of distinct paths of duration n seconds which the counter may take to reach F from S. Prove that for $n = 1, 2, 3, \ldots$ $a_{2n-1} = 0$, $a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$, where $x = 2+\sqrt{2}$, $y = 2 - \sqrt{2}$.

*SF3 Show that for any vectors $\underset{\sim}{a}, \underset{\sim}{b}$,

$$|\underset{\sim}{a} \times \underset{\sim}{b}|^3 \leqslant \frac{3\sqrt{3}}{8}|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2|\underset{\sim}{a} - \underset{\sim}{b}|^2$$

<u>*Alternative.*</u> S is the area of a parallelogram $OABC$. Prove

$$\underset{\sim}{S}^3 \leqslant \frac{3\sqrt{3}}{8} \underset{\sim}{OA}^2 . \underset{\sim}{OB}^2 . \underset{\sim}{OC}^2 .$$

GDR1 Given $x_1, x_2, \ldots, x_n$ $(n \geqslant 2)$ real numbers with $x_i \geqslant \frac{1}{n}$, $(i = 1, 2, \ldots, n)$, and with $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$, find whether the product

$$P = x_1 x_2 x_3 \ldots x_n$$

has a greatest and/or least value and, if so, give these values.

GDR3 R is a set of exactly 6 elements. A set F of subsets of R is called an 'S-family over R' if and only if it satisfies the three conditions

(1) For no two sets X, Y in F is $X \subseteq Y$,

(2) For any three sets X, Y, Z in F, $X \cup Y \cup Z \neq R$,

(3) $\bigcup\limits_{x \in F} X = R$.

$|F|$ is the number of elements of F (i.e. the number of subsets of R belonging to F). Determine, if it exists, $h = \max |F|$ the maximum being taken over all 'S-families over R'.

*GR1 Show that $\frac{20}{60} < \sin 20^0 < \frac{21}{60}$.

*GR4 Find all systems of logarithms in which a real positive number can be equal to its logarithm, or prove that none exist.

IS2 The <u>non-negative</u> real numbers $x_1, x_2, x_3, x_4, x_5, a$ satisfy the following relations :

$$\sum_{i=1}^{5} i x_i = a$$

$$\sum_{i=1}^{5} i^3 x_i = a^2$$

$$\sum_{i=1}^{5} i^5 x_i = a^3$$

What are the possible values of a ?

IS4 Let K denote the set $\{a,b,c,d,e\}$. F is a collection of 16 different subsets of K and it is known that any three members of F have at least one element in common. Show that all 16 members of F have exactly one member in common.

ND1 Inside an equilateral triangle ABC one constructs points P, Q and R such that

$$\angle QAB = \angle PBA = 15^0,$$
$$\angle RBC = \angle QCB = 20^0,$$
$$\angle PCA = \angle RAC = 25^0.$$

Determine the angles of triangle PQR.

PL1 Let there be given m positive integers $a_1,\dots,a_m$. Prove that there exist less than 2^m positive integers $b_1,\dots, b_n$, such that all sums of distinct b_k's are distinct and all a_i ($i \leqslant m$) occur among them.

R1 Consider the sequences (a_n), (b_n) defined by $a_1 = 3$, $b_1 = 100$,

$a_{n+1} = 3^{a_n}$, $b_{n+1} = 100^{b_n}$. Find the smallest integer m for which

$b_m > a_{100}$.

S2 Given the integer $n > 1$ and the real number $a > 0$, determine the maximum of

$$\sum_{1}^{n-1} x_i x_{i+1}$$

taken over all non-negative numbers x_i with sum a.

SU1 Let N be the number of integral solutions of the equation $x^2 - y^2 = z^3 - t^3$ satisfying the condition $0 \leqslant x, y, z, t \leqslant 10^6$, and M be the number of integral solutions of the equation $x^2 - y^2 = z^3 - t^3 + 1$ satisfying the condition $0 \leqslant x, y, z, t \leqslant 10^6$. Prove that $N > M$.

SU3 There are two circles on the plane. Let a point A be one of the points of intersection of these circles. Two points begin moving simultaneously with constant speeds from the point A, each point along its own circle. The two points return to the point A at the same time.

Prove that there is a point P on the plane such that at every moment of time the distances from the point P to the moving points are equal.

USA4 Find all natural numbers n for which $2^8 + 2^{11} + 2^n$ is a perfect square.

USA5 Circle O with centre O on base BC of isosceles triangle ABC is tangent to the equal sides AB, AC. If point P on AB and point Q on AC are selected such that $PB \times CQ = (\frac{1}{2} BC)^2$, prove that line segment PQ is tangent to circle O, and conversely.

USA6 Given a point P in a given plane π and also a given point Q not in π. Show how to determine a point R in π such that $(QP + PR)/(QR)$ is a maximum.

YU4 Prove that the functional equations $f(x+y) = f(x) + f(y)$ and $f(x+y+xy) = f(x) + f(y) + f(xy)$ $(x,y \in \mathbb{R})$ are equivalent.