THE XXI INTERNATIONAL MATHEMATICAL OLYMPIAD
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PROBLEMS PROPOSED BY THE U.S.A.

USA1 If $a_1, a_2, \ldots, a_n$ denote the lengths of the sides of an arbitrary n - gon, prove that

$$2 \geq \frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \cdots + \frac{a_n}{s - a_n} \geq \frac{n}{n - 1},$$

where $s = a_1 + a_2 + \ldots + a_n$.

USA2 From point P on arc BC of the circumcircle about triangle ABC, PX is constructed perpendicular to BC, PY perpendicular to AC and PZ perpendicular to AB (all extended if necessary). Prove that

$$\frac{BC}{PX} = \frac{AC}{PY} + \frac{AB}{PZ} .$$

USA3 Given $f(x) \leq x$ for all real x;

$f(x + y) \leq f(x) + f(y)$ for all real x, y;

Prove $f(x) = x$ for all x.

USA4 Find all natural numbers n for which $2^8 + 2^{11} + 2^n$ is a perfect square.

USA5 Circle O with center O on base BC of isosceles triangle ABC is tangent to the equal sides AB, AC. If point P on AB and point Q on AC are selected such that $PB \times CQ = (\frac{1}{2} BC)^2$, prove that line segment PQ is tangent to circle O, and conversely.

USA6 Given a point P in a given plane π and also a given point Q not in π. Show how to determine a point R in π such that $(QP + PR)/(QR)$ is a maximum.