39th Balkan Mathematical Olympiad, Agros, Cyprus, May 4 - 9, 2022

A report by Robin Bhattacharyya (Leader of the UK team)

By the spring of 2022 it was finally, and thankfully, becoming possible again to meet up in person for mathematical training camps and competitions, both in the United Kingdom and internationally. Thus, we were able to send a team out of the UK to travel to a Balkan Mathematical Olympiad for the first time since 2019, and we were all very pleased indeed to be going.

(The UK had competed in the 2020 and 2021 competitions, but remotely – and both these contests were held in the Autumn, rather than in the conventional window of April 25 – May 10.)

Selecting the UK team

We start the selection process for our UK team in schools, with a 90 minute long multiple choice test called the Senior Maths Challenge, which is held in November, for about 80,000 people; of these, about 1000 qualify for the British Mathematical Olympiad Round 1, a three and a half hour exam taken in schools in December, then marked online by volunteers. Approximately 100 qualify from this for the British Mathematical Olympiad Round 2, this academic year held in January, again in schools, and actually marked in person now. After this, 24 students were chosen for our six day long March/April residential **training camp, back in person at Trinity College, Cambridge** this year after two years on Zoom. There were two selection test exams here (each lasting for four and a half hours), and at the end of this camp we selected a team for the Balkan Olympiad.

For many countries, their Balkan Mathematical Olympiad team is practically identical to their International Mathematical Olympiad (IMO) team, but this isn't as true for the UK. We use this competition partially for training for the future, and also to give more people experience of real competitions, so **anyone who had previously represented their country** in the International Mathematical Olympiad (IMO) or in the Balkan Olympiad **was not selected** for the UK team (with one exception – as she'd taken the Balkan Olympiad online in a year in which the UK students didn't even leave their own homes to compete).

The UK team selected was:

Benjamin Gillott	Collingham College, London
Thomas Kavanagh	King's Maths School, London
Hayden Lam	Tonbridge School
Sida Li	Reading School
Eleanor MacGillivray	King's, Ely
Lingde Yang	The West Bridgford School, Nottingham

The three adults accompanying the students were:

Robin Bhattacharyya   Trent College   (Leader)
Ina Hughes   Open University   (Deputy Leader)
Jamie Bell   University College, London   (Observer with Leader)

Ben and Eleanor were in their final year at school; the other four students were in Year 12.

The UK first entered the Balkan Olympiad in 2005, as a guest country, but the competition had begun in 1984, and this year's was the 39th contest. Jamie had been a student in the UK team at the 2016 Balkan Olympiad in Albania; this was my fifth Balkan Olympiad, as a Leader or Deputy Leader, but the first time actually travelling to a competition for a while (since my sons were born).

The 11 member countries take it in turns to host, and this year the competition was held in Cyprus, and it took place from Wednesday 4th May until Monday 9th May; it was based inland, among the mountains in the pretty town/village of Agros.

The Competition Itself

Tuesday 3rd May – Meeting Up in the UK

My guide book says that 42% of tourists to Cyprus come from the UK, and it was pleasing but unsurprising to find that there were many direct flights available for us to choose from each day. We later found out that many other teams, though based closer to Cyprus than us, had much longer and more complicated journeys than us.

We chose to fly out just before 9am, so that we could get to our hotels in Cyprus at a sensible time. Therefore, we meet up on the previous evening, Tuesday 3rd May, at the Premier Inn near Heathrow Terminal 5, with the students arriving at about 6pm. They play some cards with Jamie after checking into their rooms, then go to dinner with Ina now, as well; and I join them there a bit later, having travelled down from the East Midlands after a full day of teaching. As I'd come through St Pancras station, I'd seen people with a laminated blue and yellow flag with Cyrillic information on, waiting for the Eurostar to arrive – a sign of the times.

The six students are in good spirits, combining chatting and playing cards with trying to solve some hard maths problems, **often using the paper napkins to write on**. It's actually **Lingde's birthday** today, and we have chocolates and party hats for all, and there is also a big badge for Lingde (thank you Ina!). And it was **Sida's birthday yesterday!** Eleanor has brought along the physical book of Balkan Olympiad problems up to 2013, that Dominic Yeo had sent her, for use by all of our team in the build up to the exam. I give everyone an extra present – their official blue polo shirts featuring the UKMT logo and the name and location of the event (and thank you also to my Year 12 Further Maths students at Trent College for their advice about what style and colour of shirt that like-minded people of their age might wish to wear!). Thomas tries on his XL polo shirt – definitely big enough! After more maths, the team clear away from the dinner table their UKMT maths books and copies of Evan Chen's geometry book, and go off to their rooms shortly after 10pm. Jamie, Ina and I discuss our roles in the upcoming competition for a bit and then retire too.

Wednesday 4th May – Departure for Cyprus

My new alarm clock wakes me at 5am. I do some last minute rearranging of things into the three bags we are each allowed on our British Airways flight (one checked in, one for the overhead compartments and one for under the seat in front) and make it down to breakfast, which opens at 5.30am. But they are surprised to see us there, and it seems that there was some misunderstanding about whether we had booked or not. So it is going to be Plan B – eat when at the airport – which is what many were planning to do anyway.

The hotel is a couple of miles from the airport Terminal, and our Hoppa shuttle bus is due at 6.01am. Everyone is up and packed and ready to go, and soon we are at Terminal 5, a modern and striking building (though slightly confusing to navigate).

There is some redistribution of liquids and compasses into checked luggage from Ben, who isn't checking anything in, and we are through bag drop and then security (slowly, as their system was down for a while, and things had to be done manually with stickers), and we are now ready for some breakfast. Ina is acting as mum, and organising the food. **The students talk about maths**, discussing the precise statement of the Lifting the Exponent lemma/theorem (LTE), and then learning about Zsigmondy's theorem too, and Schur's theorem. They are certainly doing their homework!

We won't be getting food on board so it is back to Pret a Manger for some sandwiches/rolls to take on, for our lunch. Our Airbus 320 has unusual sky blue livery, unlike the other BA planes – some kind of 'Better World' fuel efficiency climate change concern partnership, I believe. We sit together in three rows very near to the back of the plane. UKMT has given the adults the window seats, perhaps aware that the students would be spending all their time solving past Olympiad contest or shortlist problems. We wheel away from the mathematically pleasing Stand 512, and take off at 9.27am.

It is **a grey day in London**, and soon the only view below us is of clouds. Next to me, Eleanor is working on a combinatorics question, and then with Sida she attempts an inequality that Sida's mentor has given him. We fly over snow covered mountains in the Alps and later on over finger like peninsulas in Greece, and then we follow close to the coast of Turkey where the ground looks dry and brown, as it is when we see Cyprus. There has been some conversation about things such as publicity for Olympiads, but mostly the students have just been doing maths during the flight.

Late in the journey we are told that our flight has been **randomly chosen by the government of Cyprus for Covid testing!** Five or more days in isolation would mean no experience of the Olympiad for anyone testing positive, and when Sida works out the probability that at least one of us will test positive, based on 1 in 23 people in the UK having Covid right now, not everyone is happy. After flying right over Larnaca airport and then out to sea (which looks choppy – it is a windy day), we return and touch down at 3.36pm local time (two hours ahead of the UK), after a flight of just over four hours. We are rushed through Covid testing and passport control like VIPs, then collect our checked in bags and go out into the sun to the buses to take us to the Olympiad locations. The teams from Bulgaria and Greece have arrived earlier (the latter on two flights, some from Athens and some from Thessaloniki) and are waiting for us.

It turns out that there is entirely separate transport for people going to the Leaders' site and those going to the main site in Agros, so Jamie and I quickly say **goodbye to Ina and our six students**, and we get into a taxi van to Limassol. Alex from Greece and Stanislav from Bulgaria are already there, with their **masks on, as the law in Cyprus decrees for indoor public spaces**. We have a journey of about 50 minutes, mainly on a motorway. The UK was in charge of Cyprus from 1878 to 1960 and

our influence remains in various ways, for example in the three pronged plugs, and also driving on the left.

We arrive at our upmarket hotel (Ajax) at about 5.30pm. We have our NHS Covid Passes scanned as we enter (this is the law here for hotels and indoor restaurants). At 6pm there is a meeting of the team Leaders in the 'Jury Room' (the Iphigenia Room of the hotel) to collect our convenient Cyprus Mathematical Olympiad backpacks and Balkan Olympiad T shirts, caps with many digits of pi on, programmes of events and importantly the shortlisted problems for the competition, with solutions. Jamie and I are straight to work trying these questions for ourselves (while consuming the more-ish sweets that have been helpfully laid out in small bowls).

A series of text messages to my phone (actually the same phone I used in the 2009 Balkan Olympiad) reveals that we're **all in the clear for Covid**. Communication of this fact to Ina is allowed, so that the team can relax about that. I find out later a bit about what they've been doing; their journey had been via a different route, going inland and near to the capital Nicosia and then across to Agros, on a bus with some friendly and enthusiastically noisy Greeks.

Dinner for us is at 8pm, a satisfying buffet in the restaurant, which is right opposite our Jury Room. We have a brief official meeting scheduled at 9pm, but basically it's back to work for Jamie and me, on the shortlisted problems again, of which we solve a couple and make serious starts at several others. There are **21 problems, submitted by various countries**. Some are from the UK – Aron, Daniel G, Dominic Y, Lex, Sam, and Tommy have been busy!

At 10pm we relocate to the hotel bar. Like the restaurant, it is large and very smart, but it's early season and the bar is practically empty. Peanuts are provided, with our glass of Cypriot wine / G & T. We discuss the problems some more, especially a tough but appealing algebra question. The TV alternately shows basketball and the Champions League football semi-final between Real Madrid and Manchester City. We go to our rooms with Manchester City two goals up on aggregate and about ten minutes to go; it seems safe to leave at that point - but it turns out that wasn't the case.

Thursday 5th May – Leaders Select the Exam Problems

After breakfast, we have a meeting of the Jury at 8.30am - the Jury consists of the Leaders of the teams, and any Observers with Leader (such as Jamie). There is one vote for each member country; we are a guest country, so we don't have a vote, but Jamie and I can join in the discussions. The imposing figure of the Chair of the Jury is Gregoris Makrides, and Problem Selection Captain Demetres Christofides is a key figure too. Demetres is well known to two of the staff from our Cambridge training camp (from a month earlier) – his PhD supervisor, Imre Leader, and fellow PhD student from his time in Cambridge, Paul Russell. Demetres talks through the problems and suggests their difficulty rating; there aren't too many comments about this from the jury. By 9.20am the official difficulty ratings have been set (easy or easy-medium or medium, etc.).

Assylbek tells me that he took 20 hours to arrive from Kazakhstan, flying from Almaty to Frankfurt, and waiting there for a while before heading to Cyprus. Serbia also came via Frankfurt, even though that's in completely the wrong direction for Cyprus. Both countries have had a journey of more than twice the length of a direct (great circle) trip, and they're not the only such teams here.

After some more time to consider the submitted problems, there are a couple of proposals to delete a couple of questions from the list, in one case because it might be too easy, and in another case because it might be similar to a problem that someone [Catalin (the Romanian Leader)] had seen

before. These comments are certainly noted, but nothing will be deleted, Gregoris tells us. It is, though, Demetres who is in charge of most of the discussions. [He also had the role of Problem Selection Committee Chair in last year's online Balkan Olympiad, also organised by Cyprus.]

It only takes **about half an hour to decide the problems for the competition**. Instead of choosing questions one by one, people propose whole sets of four questions (one each from algebra, combinatorics, geometry and number theory), to be the set of questions for the four and a half hour long exam paper. Greece is first to propose (a set which later becomes known as Proposal A). Then Bosnia (with Proposal C), Kazakhstan, Serbia and Albania. Demetres is asked for his views, and he offers his selection, which is very similar to Proposal A.

In the end there are eight proposed sets, and the member countries will vote on them. They can vote more than once on each round of voting, and the least popular choice drops out each time, with two choices to go out the first time. The votes are by a show of hands, some of which take a while to go up, as things are considered. At one point, by accident one of the guest countries attempts to vote, but that's not allowed. At 10.15am we're down to two proposed sets of questions (with only one question in common): Proposals A and C. One member country (Turkey) is absent this year, so there are ten countries voting, and the vote is intriguingly tied at 5-5. We wonder what the rules are. Someone suggests asking the guest countries – and these (including the UK) all prefer the same set; the Chair, Gregoris, says that Regulation 7 gives him the **casting vote**, and he votes the same way as the guest countries suggest: Proposal A is chosen for the contest.

Only now do we find out which questions come from which country. **Two of the selected questions are from the UK** (actually Jamie and I already knew this): a geometry question from **Dominic Yeo** which will be Problem 1 here, and a harder combinatorics question from **Tommy Walker Mackay** which will be Problem 4. The other two questions are from Greece: a number theory question for Problem 2, and a functional equation to be the 'algebra' question, for Problem 3; I think I'll be impressed if anyone from the UK team can solve the latter.

Now it's time to finalise the English language version of the problems. This is partly to make it into something that can as far as possible be translated word for word into each other language – as students will take the contest in their own language – and partly for maximum clarity, so that in the first half hour of the contest when written queries about the problems are allowed to be submitted by students, we are not inundated by questions. Examples of changes made here are 'YZ bisects AC' changing to 'the line YZ passes through the midpoint of the line segment AC', and $a^{2021}|n$ changing to 'a^{2021} divides n'. In the third problem, an extra set of brackets is inserted in one part of the equation around f(x), for extra clarity. And in the fourth problem, which had quite a few sentences in it, the wording is picked over, partly to deal with the case where a frog is already on a particular square (which we now say is equivalent to a frog reaching that square in zero hops).

The English language version is getting longer, but still fits on one page. The line spacing is changed, though, to make more room, so that if a translated version gains more words, it too can fit on one side of A4 paper. There is further discussion about the wording of Problem 4, and it amuses some that there is more to change to the English language version of something submitted by the United Kingdom than to something submitted by another country. By 11.40am we are finished. The problems are made available in TeX for those who use that (most countries do), so that they can start translating immediately, and those that need a version in Word will have to wait a little more time. A spare laptop is available for anyone who needs it to use in translation, but I don't think anyone does. Of course, Jamie and I don't need to translate anything; Problem 4 even says 'colour' (rather than 'color')!

The Exam Paper

Here are the four exam questions for this year's contest, in their final form:

1. Let ABC be an acute triangle such that $CA \neq CB$ with circumcircle ω and circumcentre O. Let t_A and t_B be the tangents to ω at A and B respectively, which meet at X. Let Y be the foot of the perpendicular from O onto the line segment CX. The line through C parallel to line AB meets t_A at Z. Prove that the line YZ passes through the midpoint of the line segment AC.

2. Let a, b and n be positive integers with $a > b$ such that all of the following hold:

    (i) a^{2021} divides n,
    (ii) b^{2021} divides n,
    (iii)   2022 divides $a - b$.

    Prove that there is a subset T of the set of positive divisors of the number n such that the sum of the elements of T is divisible by 2022 but not divisible by 2022^2.

3. Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ such that
    $$f\left(y(f(x))^3 + x\right) = x^3 f(y) + f(x)$$
    for all $x, y > 0$.

4. Consider an $n \times n$ grid consisting of n^2 unit cells, where $n \geq 3$ is a given odd positive integer. First, Dionysus colours each cell either red or blue. It is known that a frog can hop from one cell to another if and only if these cells have the same colour and share at least one vertex. Then, Xanthias views the colouring and next places k frogs on the cells so that each of the n^2 cells can be reached by a frog in a finite number (possibly zero) of hops. Find the least value of k for which this is always possible regardless of the colouring chosen by Dionysus.

Time: 4 hours and 30 minutes
Each problem is worth 10 points

I later find out some more of what our UK students had been doing up to this point. After their journey of an hour and a half or so from Larnaca airport to Agros, in the Troodos mountains region (on the day before), Ina had helped them check into the Rodon Hotel – Rodon meaning Rose, as this small town/village is known for its rose petals, and is indeed just a couple of weeks away from its famous annual rose festival. Then the students were officially registered upstairs for the competition, before settling in. UKMT (and other) playing cards had been brought along, and were very popular. Now this morning they had spent a good while trying a number theory question they simply referred to by its IMO shortlist code of 'N8' – apparently Sida solved it – before several of them went on a walk in the surrounding area, which is very rural.

Now back to the Leaders in Limassol…
Translation isn't taking particularly long, at least for most. I chat to Valeriu Guţu from Moldova. He'd been at the first Balkan Olympiad that I'd been to, in Romania back in 2005. And I'd met him again in

2009, in the UK, at the Team Maths Challenge Final in London. I'd mentioned then that I would be passing through Moldova later that summer and he suggested we meet up there – which we did, and he very kindly showed me around. I'd been travelling from Odesa on that occasion; thirteen years on, that part of the world is a much less happy place. It seems clear that things won't be forgotten, or forgiven for a very long time.

Valeriu mentioned how there are many fewer students of maths at university in Moldova than there used to be a couple of decades ago, as people are attracted to other subjects, and many of the country's brightest students go off to university in other countries.

Massimo from Italy tells us of the 'Italian Paradox' – with no training camps because of Covid, Italy nevertheless had its best IMO results in the last two years (perhaps students had little else to do but work at home, on their own, on Olympiad maths).

Lunch is at 1pm in the hotel. With much of the work done, we have a glass of wine with our food. Translations are to be printed out for inspection by all the Leaders and Observers at 3pm, but without any translation to do ourselves, Jamie and I take a walk to the beach, which is about 10 minutes away. It's good to see the sea, one of the main attractions of Cyprus. It's actually quite windy today. We come back for the meeting, at which most of the translations appear. Afterwards, Jamie has a 20 minute swim in the outdoor pool. And then rain starts to fall!

The Opening Ceremony

We are taken by bus to the Laniteio Lyceum, a school in Limassol where the Opening Ceremony will take place. While we wait before the ceremony we have snacks in a room full of trophies; Demetres warns us not to eat too much, as there will be a big dinner later, at a restaurant.

We sit in the second row for the ceremony, which runs from 6.25pm to 7.20pm. It begins with some very good singing and music from local students. Then there is some traditional dancing, in costume. Gregoris Makrides speaks about the school of the future, about maths for technology, and a relevant summer school in Cyprus. There are speeches on behalf of the Mayor of Limassol and the Minister of Education. Then **the teams parade on the stage with their flags**. Italy have with them balloons in the colours of their flag; they are clearly a popular team (having made a positive impression already in Agros). Azerbaijan are not here yet; it's proving a difficult journey to Cyprus for them.

After the ceremony, the Leaders see something of the modern centre of Limassol from our bus, and then on foot too as we travel at a leisurely pace to our **outdoor restaurant in the old town**. The students had had some time in the old town before the Opening Ceremony, having left Agros at about 2pm and arrived here at about 3pm. Now they are travelling by bus back to Agros, where they will have their dinner before retiring for the night, with the exam tomorrow.

We Leaders walk through the Old Port and Marina area, with its yachts and piers. Then there are indeed many courses of traditional food at the restaurant. Catalin, who organised the online Balkan Olympiad in 2020, is the Romanian Leader; he tells us about his passion for tennis (very recently he was in Belgrade watching a tournament) and for FC Barcelona. Nikola from North Macedonia tells us about the team selection process and training in his country, run by quite a small number of people.

We're back at the hotel shortly after 10pm and are straight into a meeting to formally approve the translations. It's interesting to see the different scripts and languages. On Problem 4, the individuals are called Dionysus and Xanthias, named after characters in The Frogs by Aristophanes. Most countries have kept these names, but one went for Ana and Maria, and another for Bojan and Lena. We can't work out what names the Arabic version used. Hasan finally arrives from Azerbaijan. His

team have had three flights, and connection delays, and it's taken them 20 hours to arrive. (It was even more complicated for Hasan himself, as he had also had to travel first to collect a visa, meaning seven flights in two days!). His reward is to be given the English language script for translation to be completed by him that night.

Friday 6th May – Exam Day

In the morning we pack up, as all Leaders and Observers are from now on to be based in Agros. I enjoy the fresh lemonade at breakfast. We'd been told that the bus would leave at 8am, and anyone arriving after 8.05am would have to take a taxi. We pull away at 8.13am. The last Leader comes by car with Gregoris. It's a journey of one hour, mostly on bendy **mountain roads**; Agros is at an altitude of 1100m or so, higher than any point in England (but not Scotland). The hotel is above the village, and has splendid views. We are led in to inspect the exam room at about 9.30am. The students are all in masks and awaiting the start.

After the start, at 9.45am, the students have half an hour to submit written questions. Only nine such come in. One asks about whether it can be assumed that a certain geometrical point is the 'dumpty point'. The answer from the Jury is that 'all well known theorems/results must be defined/explained and/or cited but not necessarily proved'.

The Leaders continue their meeting, in a rather chatty room. The rough paper that students use (known here as scratch paper) can be requested by Team Leaders when marking, but will not be given otherwise. The students are writing on 'carbon paper' (carbon free, though) and the Leaders will get the top version to look through for marking, while the Cypriot experts will be given the copy.

Some countries use this Olympiad as a TST (Team Selection Test for the IMO) and some have other students back home ready to take the same exam today with a time delay of half an hour or so.

From 10.30am until 1.30pm, while the students continue to work in the exam, we collaborate to **produce mark schemes**, led by Demetres again. First we deal with Problems 2 and 3. Jamie notices that the official Lifting the Exponent proof of Problem 2 wasn't careful enough about what happened if 3 divides both a and b, and he tells Demetres about this, just before a break which is scheduled so that we can all check into the hotel. Demetres is happy to report a bit later that that could be fixed pretty easily. On Problem 3 we decide that saying the function is strictly increasing is not obvious enough from the functional equation not to require a proof. Problem 4 can be done by induction but it's a long and awkward proof and not the best way to go. Finally, we look at Problem 1, trying to be a bit generous in the mark scheme, as it's the first question.

We have a few minutes outside the hotel, enjoying the views of mountains, trees and the white houses of the village, and the cooler mountain air. Then it's time to speak to the UK students after their exam.

We see Ben and Lingde first. Ben feels the exam went well for him, while Lingde doesn't think he's managed any whole questions. Next we find Thomas, who is reasonably satisfied. Sida and Hayden have had mixed experiences, better for Sida. And Eleanor is not sure how she's done.

With Ina too, there are 9 of us together for the buffet lunch downstairs. Afterwards, from 3.30pm to 5pm, Jamie and I grill the students on exactly what they'd each done in the exam, to help us with our marking. The hotel has large lobby/lounge areas, which makes it a very good place for teams to gather, and to meet other teams.

From 6.15pm the students' scripts are available for collection by Leaders. Jamie and I have much to do, as we must look through the UK scripts for each of the four questions, but also Problem 1 and Problem 4 (the UK submitted questions) for both Cyprus A and Cyprus B, where we will have to act as the local co-ordinators. We look through some scripts, then it's dinner downstairs together with Ina and the team, and then it's back up to the lobby for much more analysis of scripts. **We're still going after midnight when five of the UK team find us**. They discuss solution attempts, in particular Eleanor on Problem 3 – this is not quite the most direct approach, there are subtle issues over limits, not quite everything is explained on the page, but Jamie sees that all the big steps are there.

Jamie and I are done by about 3am. Not bad – it's been later at some previous competitions!

Saturday 7th May – Deciding the Marks

Yet **another birthday today** – **for Thomas**. Indeed, three of our team were born within three days of when I went off to my own first Balkan Olympiad, in May 2005.

After breakfast there's a bit more chance to look at scripts before the scheduled co-ordination meetings begin. We check through the UK geometry scripts again.

Some Comments on the Exam Problems

Now might be a good time to briefly discuss the competition problems…

In **Problem 1**, as in many geometry questions, one would be well advised to look at angles, and search for similar triangles and cyclic quadrilaterals. The point Y seemed to me a bit awkward to deal with, at first. Does it lie on any circles? Considering C, Z and A together, and then also with Y, is useful. This question can be solved without any special knowledge. But the argument can perhaps be made shorter with some extra background knowledge, such as of symmedian lines.

In **Problem 2**, where could one start? There are potentially very many divisors; how do we choose some? If you realise that a^{2021} and $a^{2020}b$ and $a^{2019}b^2$ and so on must all divide n, then these could form your set. They can be summed into a simple form (for example as a geometric sequence), but we must then prove that 2022^2 does not divide the sum. To do this, we can use the Lifting the Exponent lemma/theorem. But if 1011 divides n then we can't use 3 or 337 in the Lifting the Exponent lemma - so a different argument is needed in this case, and in fact we just need to consider n = 1011 then; and in this case a simple suitable set can easily be found.

In **Problem 3**, substituting in values and working from there doesn't itself seem to solve the problem. It's not a long argument to show, from the functional equation given in the question, that the function f(x) must be strictly increasing. Thus, proving that f(x) = x for the positive rational numbers will be enough to deduce that f(x) = x for all positive real numbers x. And if we can prove that f(1) = 1 then in fact it follows pretty quickly that f(x) = x for all positive integers, and then for all positive rational numbers. So the key is to prove that f(1) = 1. Separate arguments can be used to exclude the possibilities that f(1) < 1 and that f(1) > 1. The harder part can be done by a limiting process, using induction to get a formula in terms of n, and then allowing n to tend to infinity, and looking for some contradiction in the sizes of things; it seems to me quite demanding.

In **Problem 4**, the key is to think about the regions of one colour, and in particular about the boundaries between such regions - though there are many things one might try in this problem before a careful count of vertices and the boundary pieces that they are in is embarked upon. The

number of separate boundary chains/loops is key. More precisely, consider the number of connected components of the graph having as vertices the grid points which are corners of cells, with the graph created by drawing a graph edge between two vertices if and only if the vertices are adjacent corners of the same grid cell and the grid edge between them has differently coloured cells on either side. This number is very closely related to the number of frogs needed. And then counting all the corners of the cells of the grid and considering the smallest possible number of vertices in each connected component of the graph, we get a bound which is tight, when the right regular example of an arrangement is found. It looks as though it's not the most complicated argument, but as I said, there are many other approaches which seem more obvious to try, but which don't work.

[We later discover that essentially the same question had appeared in competition before, in 2018 (e.g. in the Danube Competition in Mathematics), but this was entirely coincidental; none of the Leaders or students knew this, and this just reflects that with so many competitions happening, it is inevitable that the same good ideas occasionally occur to different people, in different countries.]

Co-ordination

Four of the UK team have clear solutions to Problem 1; Thomas uses properties of harmonic quadrilaterals, all done correctly. It's Hayden on Problem 2 and Eleanor on Problem 3 that we're keen to look at again. But for now, it's our **first co-ordination session**. These are each 20 minutes long, and are a discussion between the adults with each team and the Cypriot experts on that question, to decide the marks for that country's scripts on that problem. We will have four of these sessions for the UK team (one for each question), and also four sessions scheduled for us for the Cyprus teams, on Problems 1 and 4.

First we have Cyprus A for Problem 4, at 9.20am. We are asked if we could co-ordinate Cyprus B at this point too; this is fine for us. It's tough for a student to get more than one mark on this question – for example, trying induction doesn't get any credit unless there is a detailed and accurate statement of what is being claimed, and some care taken; there is no easy induction proof, and it's proving to be tough to find any correct solution. After a break we're back to deal with the UK on the same question. We explain what Ben and Hayden have each tried, but accept that these arguments do not lead to a solution. Eleanor's idea is described by the co-ordinators as a good one, but again it doesn't seem to lead to a solution; five of our students get one mark each, for a diagram and the formula for the 'best case' scenario, and that's it.

At 11.40am it's time to go through the geometry, with Cyprus first. These days people know many facts and methods that can be applied in geometry – both the Cyprus students and the UK students – and some of them use symmedian properties of harmonic quadrilaterals, dumpty points (with dumpty written in these Roman letters, in amongst a Greek language explanation), cross-ratios and theorems of Newton and Maclaurin. One Cyprus B student has proved rather more than what the mark scheme awards 3 marks for; that student gets 5 marks. There are also several full solutions from the Cyprus teams, which we award 10 marks each, after in some cases we ask for translation from Greek to English to help us to check. Then it's time for the UK geometry scripts. We carefully talk through the harmonic quadrilateral method from **Thomas; it's fine, as are the solutions from Ben, Sida and Eleanor**. We have a total of 40 marks for this question. (Hayden and Lingde didn't make much progress towards a solution.)

We tell our students (some of whom have been doing sport) how they've done, before lunch.

Eleanor can be seen in the hotel lobby with her earphones in as she speaks on her phone to another UK student, Mohit, who isn't here in Cyprus. Mohit offers his opinions on her effort on Problem 3.

No effort is being spared to try to get our marks: thank you to Mohit for helping from thousands of miles away!

Jamie and I should be in for co-ordinating Problem 2 at 3.20pm, but they are running late – for some teams, 20 minutes has not been enough. Indeed, when we do go in, we also have plenty to talk about, though **Ben, Thomas and Sida have written up their solutions very nicely and clearly, and all get 10 marks**. We suggest 4 marks each for Eleanor and Lingde. They each have correct examples, but can't prove all the required properties in some cases, notably when b = 1. This isn't too surprising, as without Lifting the Exponent this doesn't seem to work, and the proposed official solution that avoided LTE had actually failed on this point too. After some time checking that the examples were there in the different cases, both do get 4 marks. Hayden was rushed when writing up, and his handwriting is really hard to decipher. I read out his lengthy script word for word, with help from Jamie. We explain that several applications of the binomial theorem have been used instead of LTE, and then there is plenty more to explain too. The co-ordinators are curious about how the case where a is a multiple of 1011 is treated by Hayden. They note that he has actually dealt with this, if he ignores an unnecessary coprimeness restriction which he'd introduced. After some time, it is agreed that **Hayden's script is worth 10 marks**.

I am very pleased with how our team have fared on Problem 2. They've earned 48 marks out of 60, which is **an excellent result**.

Soon after, we are called into the Problem 3 session, slightly early. There is a mark available on this question for showing the important fact that f(x) is strictly increasing; most of our students get this mark, and Sida gets another mark too, for proving a bit more about f(1) versus 1. Jamie explains how Ben had proved the full continuity of the function f(x), in rough (in neat he'd shown continuity from above). Ben has made good progress and there is another student (from Greece) who had a similar argument that went onto being a full solution. The co-ordinators explain that in this circumstance they are offering 4 marks. We wondered if it might be worth a bit more, but it isn't a complete solution, and the mark scheme doesn't allow any more.

Finally, it's **Eleanor's script**. Jamie takes the lead on this, and explains that despite a bit of missing algebra to bound the ratio of f(x) to x, and the strictly increasing nature not being explicitly proved, the solution does work, using an induction similar to the one in the official solution, and using bounding arguments, though Eleanor is trying this for general x rather than just for x = 1. **The script earns 8 marks**, and this is our last bit of co-ordinating at this Olympiad.

The co-ordinators for this question have a chat with us afterwards, this having been their final co-ordination session too. One of them was unlucky enough to miss the IMO as a student himself when it was cancelled in 1980. He and another co-ordinator have worked in Manchester (at universities there), and we talk about the UK and what's happening there; clearly there are links between Cyprus and the UK. We also talk about the situation in Cyprus, with tourism diversifying, for example away from the Russian market this summer; and there are concerns over dependence on Russian capital.

At about 5.30pm we are able to tell our team what marks they've got – though the medal boundaries haven't been decided yet. There should be the final Jury meeting at 6pm for that purpose, but Problem 2 co-ordination is still going on, and will be going on for a while; the meeting is postponed until 9pm, after dinner.

Co-ordination has gone smoothly for us, and we are happy with the marks awarded. Eleanor is very pleased with everyone's efforts on her script for Problem 3, and Hayden wasn't sure that he'd get full marks on Problem 2.

Deciding Medal Boundaries

At 9.05pm we're into the Jury meeting. It's quite busy here in the ('Apollo') room – I count 33 people present (with Senada the Bosnian Observer being the only woman). Some time is spent resolving a request for an extra mark on one script for Problem 2, with the dispute being between Albania (whose Leader is Adrian) and the co-ordinators. The Leaders of the member countries vote for the mark to be awarded (no-one votes against, though there are some abstentions).

With this dealt with, we receive the scores of all the students, from all countries. The main business of the meeting is to decide medal boundaries, starting with the minimum mark required for a Bronze. It's quite a generous decision, by a majority vote: 12 marks will be needed for Bronze, giving medals to more than the regulation two-thirds of the member country students. But Hayden misses out by a single mark; at least he gets an Honourable Mention, for full marks on Problem 2. The Jury agrees to follow the 1:2:3 ratio of Golds to Silvers to Bronzes as closely as possible. This means that 22 marks are needed for Silver and 31 marks for Gold. Thomas has a Bronze, missing out on a Silver by a single mark.

There are **three Silvers for the UK team** – for Ben (who is equal 14th in the competition, out of the 93 competitors), and Sida and Eleanor (both equal 22nd). This is only the third time in 18 attempts that at least half the UK team at a Balkan Olympiad has got at least a Silver (the previous times being 2013, also in Cyprus, and 2020).

The top score in the contest is 38/40, by Borislav Kirilov from Bulgaria. The second highest mark is 34 by Andrei Moldovan who is Romanian. Only two students have scored more than 4 marks on Problem 4.

We have a moment of silence for a problem setter, Evaggelos Psychas, very active submitting problems for the Balkan Olympiad (and IMO) in the last 10 years; he died a couple of weeks ago.

Zenan welcomes teams to the Junior Balkan Mathematical Olympiad, which will be held in Sarajevo from June 27 to July 3. Nobody has offered to host next year's Balkan Mathematical Olympiad, so that will be decided at the next meeting of the regional mathematical society, MASSEE, on June 4 or 5. And this meeting closes at 10.15pm.

Jamie has by now told our team what their medal situation is. While card games happen in the lobby, I suspect that Eleanor is the top girl in the competition, having beaten Diana Țolu of Romania by just 1 mark (just as she did at EGMO) and I enquire of other leaders whether they have any girls with a higher score than Eleanor's – they don't, so **Eleanor is indeed the highest scoring female at this competition**.

The Results

Here are the results of the UK team:

		P1	P2	P3	P4	Total	
Benjamin	**Gillott**	10	10	4	1	**25**	Silver
Thomas	**Kavanagh**	10	10	1	0	**21**	Bronze
Hayden	**Lam**	0	10	0	1	**11**	HM
Sida	**Li**	10	10	2	1	**23**	Silver
Eleanor	**MacGillivray**	10	4	8	1	**23**	Silver
Lingde	**Yang**	0	4	1	1	**6**	

And here are the overall team scores:

Position		P1	P2	P3	P4	Total
1	Romania	60	46	44	18	168
2	Greece	60	51	30	5	146
3	Bulgaria	60	37	30	14	141
4	Saudi Arabia	60	54	17	5	136
5	Italy	60	59	4	8	131
6	Moldova	60	42	22	1	125
7	North Macedonia	60	41	14	3	118
8	Kazakhstan	60	38	16	3	117
9	Serbia	60	28	19	4	111
10	Azerbaijan	53	41	12	4	110
11	UK	40	48	16	5	109
12	Bosnia	60	35	6	3	104
13	Cyprus	43	4	4	4	55
14	Cyprus B	25	1	0	4	30
15	Albania	14	4	0	1	19
16	Montenegro	6	2	0	0	8
		781	531	234	82	1628

We note that there wasn't much of a gap between 7th place and 12th place.

It was a splendid performance by Romania, clearly the top team.

But the UK beat them (and third placed Bulgaria) on Problem 2.
Indeed, on the Problems set by countries other than the UK we came equal 5th in the competition, with one other country, but on the Problems set by the UK we were 13th.
And we were the only country to score more on Problem 2 than on Problem 1.

Kazakhstan are given their medals early, as they have to leave at 11.30pm tonight, to go to the airport. Everyone else will leave at least a day later. Most teams will leave between 11.30pm and 6.40am; only the UK, Azerbaijan and Greece are lucky enough to depart during the daytime.

There will be an excursion fairly early tomorrow morning, and the lobby is deserted by about midnight. But some of the UK team are up late upstairs, watching basketball from the NBA (Hayden, Ben and Thomas are our big basketball fans).

Sunday 8th May – Excursion and Farewell Party

We have breakfast before 8am, as it's an early start. There are three buses for the excursion: the UK will be together with Azerbaijan, Italy and Saudi Arabia. We set off at 8.30am, back down the mountain roads to the coast and Limassol, and then straight onto the **archaeological site at Kourion** by 10am. This is actually a trip to UK territory, as Kourion is on land that is part of a UK base – we held onto two military bases after granting Cyprus independence in 1960, and despite almost pulling out in 1974 we're still there (partly to assist the US), which perhaps not everyone in Cyprus thinks is the best thing.

Kourion was an ancient Greek city, built at the top of a hill for protection; the views over the sea on this very sunny day are striking. There is a restored Greek/Roman theatre here, used now for various performances (not the animal fighting that the Romans apparently had here), and the teams explore

it. We also see the remains of a house of a significant Roman, where we see what were the (communal) baths and some fine mosaics, from early Christian times.

Next it's into **Limassol itself**, the second city of Cyprus. We arrive at about 11.20am. We are encouraged to visit one of the inviting cafes in the Old Port and Marina area, and have a frappé, and we wander around for a bit past the yachts, but **our team decide they'd like to visit the beach** instead. After about a fifteen minute walk east, along the coast, we are at the first bit of beach. Ina finds a chair and orders herself a frappé. The UK students paddle in the sea and practise stone skimming. I am greeted by Zenan, who walks past me in his trunks – the entire Bosnian team have come very prepared and have gone for a proper swim in the sea. (And tomorrow they will leave the hotel early, at 6.30am, to give them a few hours at the beach near the airport before departing!)

We have to go now, as there isn't much time allowed in Limassol, and we'll have to get back to Agros for lunch. Massimo the Italian Leader is unsurprised that the Italians are a bit late back onto the bus. It's a big day for them, as the results of the Italian National Olympiad were broadcast while we were on the bus on the way out - their students here had taken an Italian exam in the afternoon after taking the Balkan Olympiad in the morning, on the same day!

We're back at the Rodon Hotel by 2.15pm, and go to lunch. We speculate on exactly what the farewell dinner/party will be like. After lunch, there is some free time. Two of the North Macedonian team, Jan and Aleksiy, ask me about the UK training and selection process, and about universities in the UK, such as Cambridge, and we talk about areas of Olympiad maths that different countries find difficult, including combinatorics in their country.

The indoor pool is out of order, but it's possible to swim outside, and Jamie (who has also gone for a few runs while in Cyprus) tries this out. It's certainly the right weather today for this.

Closing Ceremony and the Final Evening

At 6pm we have the awarding of medals. The ceremony begins with a short video by the Cyprus tourist board. Then there are some speeches, including one by Dr Philippou on behalf of the Minister of Education. Gregoris Makrides talks about students here bringing all their knowledge and experience together to solve problems in the relatively short period of four and a half hours available (he gives an analogy of an expert fixing a large ship with a simple hammer blow, and earning his fee by having had the experience to know where to knock).

The 69 medallists go up, one by one and with their flags, to be awarded their medals, in reverse score order; except, that is, for Kazakhstan who have left already. Bosnia have a huge flag. There are cheers for the sociable Greeks and for the local Cypriots. Five girls have won medals, including Eleanor's roommate here, Fidan from Azerbaijan. Our medallists go up to the front and three of the four get the flag the right way up. We're all in our blue UKMT team polo shirts. Romania are easily spotted in their very bright red polo shirts. But the Saudis are probably the most striking team, in traditional white, and with headdress.

We're finished by 7pm. We head outside, and I take photos for the Romanian and Moldovan teams of them together, before doing something similar for the UK team, who then invite me to join them in a card game I haven't played for over 25 years (and might not play again for another 25!). They seem to enjoy themselves. After quite a while I escape, as it's too dark to play properly outside. At just before 8.30pm I hand out the Participation certificates I've been given for everybody, and also Hayden's Honourable Mention certificate, and I return the exam scripts to each student in the UK

team. I have also been given the medals and certificates from the 2021 Balkan Olympiad, which Cyprus organised, but remotely.

We go in for the Farewell Dinner, another buffet, but with a difference, as a DJ starts playing some loud music, much to the delight of the youngest member of the Greek delegation, the younger brother of one of their team, who hits the dancefloor. The UK adults enjoy a glass or two of Cypriot wine; the UK team members are given mementos by some of the Saudi team.

At about 9pm the DJ puts on some older music, starting with 'Ma Baker' by Boney M – this brings some of the Cypriot organising delegation running onto the dancefloor **for some uninhibited dancing, and soon things get busy and rather wild**. The UK team all go up together, with much enthusiasm; Ben is often right at the centre of things, while Sida manages to look stylish doing the Macarena. We have an international conga line, and then a big circle with the students having their arms around each other's shoulders, while more Boney M plays ('Daddy Cool' and 'Rasputin') and also 'Cuba' by the Gibson Brothers. Jamie joins our team for a while. A couple of people are hoisted onto others' shoulders, including the Greek youngster. The Italian deputy, Nikita, manages to persuade his Leader, Massimo, to join in with the dancing.

It's great to see the students from many countries together, including the distinctive Saudis.
I pass on our friend and colleague Vesna's best wishes to Miljan, the Serbian Leader. Vesna (who now trains UK teams) has been following the progress of the Serbians on their Facebook page.

After an hour and a half of vigorous activity, things start to wind down, and by 11pm the room is empty. Many have moved upstairs to the lobby. A multi-national game of Liar's Poker ensues (I'm happy to join in). Many people have to rest before early starts, but a Greek student, two Azerbaijanis and Lingde needn't hurry, and continue the game and chat until late, with Nikita and me.

Monday 9th May – Return Journey

No hurry to rise this morning; a bit of final packing, and then breakfast a little before 10am. Demetres and another main organiser wish us well as we leave the hotel, just before 11am, for the airport. The route takes back us down the winding roads to Limassol, and then along the motorway to Larnaca airport, a journey of an hour and forty minutes. There is **loud singing in Greek for some time!** The UK retaliates with the digits of pi to the tune of Mamma Mia (see the EGMO 2022 Closing Ceremony on YouTube, after five and a half minutes – judge for yourself!). Later we hear Bon Jovi and more ABBA (Dancing Queen). The Azerbaijani team have escaped the singing duel, coming in another bus, which is towing their and our luggage. We wait a few minutes at the airport for them to arrive. Then it's goodbye to the Greeks and Azerbaijanis, though we see Fidan for a little longer.

The bag drop area is completely empty, and we are through both that and security very quickly. Lingde is well connected and seems to know exactly where our plane is – slightly late, so we'll have time for a sit down lunch, which Ina is keen for us to have. We choose a pub style restaurant. We wait a short while for our food, but Lingde assures us we won't miss our plane, and indeed when we get to the Gate, nothing happens for some time. Finally, we say **goodbye to sunny Cyprus** and board the plane, where the rules are different, and many masks come off. We take off at 3.07pm, in exactly the same numbered seats as on the way out, though this time in a typical BA livery Airbus.

After a busy few days there's a chance to rest now. I ask about motivation for getting into Olympiad problems, and Eleanor says that reading an article about Rosie Cates helped to inspire her; and going to the introductory Olympiad camp in Oxford at the end of one summer gave her confidence that she could solve hard problems. Meanwhile, Sida is writing some of the students' unofficial report of

this Olympiad. There is a question asked to Jamie about his research; he tries to explain what elliptic curves are and why they are important. There are also questions about selection (for camps, for example the winter camp in Leeds, and teams), and there are of course hopes expressed for making the IMO team. And our students study the official score list of this Balkan Olympiad in some detail.

People start comparing how many lessons they will have at school on the following day, and compete for the earliest start time and latest finish time. It's rather sad to leave the wonderful world of international competition and friendship and return to normality. We see 'Valley' painted on the seats of a London football stadium below us, but there is cloud too, and as we touch down at 5.28pm (UK time) at Heathrow it's a grey sky – and we're a little underwhelmed to be back!

By 6pm our bags have appeared, and a couple of minutes later, Eleanor is collected by her dad at Arrivals. After a bit, Hayden and Ben set out on their own. The rest of us make our way to the short stay car park, level 3, where Thomas is collected. Next it's Sida. Then Lingde too, just after 6.30pm. The three UK adults make our way to the Underground station (not that easy to find), Ina still nursing her now three wheeled suitcase, which had set out from the UK with four wheels. Jamie leaves us at Earl's Court, and Ina and I part at King's Cross/St Pancras, after she tells me about the first day and a half in Cyprus with the team – including Eleanor's desire for the perfect cuppa with the tea bags she brought from the UK, and the other students' need for warm water to drink.

I was **very impressed with our UK team**, who acted as a unit, and always turned up on time and gave a very good impression of themselves, in the exam and in everything else. It was a real pleasure to travel with and work with each of you, Ben, Thomas, Hayden, Sida, Eleanor and Lingde. You were great ambassadors, and I wish you the very best for your future in maths and in other things.

Thank you very much to **Jamie** for his efforts interpreting tough scripts and for his calm efficiency and good cheer throughout. And thank you very much to **Ina** for looking after us, entertaining us with your conversation, paying the bills, negotiating cheeky discounts, etc.

I greatly enjoyed the conversation and **banter among the Leaders** too, from the straight talking, joke telling Stanislav to the genial Zenan and Senada, and the chatty Nikola V, all smiles even before discovering the record team and individual scores that North Macedonia managed, and all the others too, including my friend Valeriu whom I first met 17 years ago, Massimo and Catalin, etc., etc.

I thank Gregoris, Demetres and the other **Cyprus organisers**, the charming co-ordinators who chatted with us after the end of co-ordination, and the Cyprus Maths Society and government for putting on a great show: a well organised and very successful return to physical Balkan Olympiads!

Thank you also to other people in the UK, especially **Kit** (and Charlie) at UKMT for helping to arrange things, our sponsors XTX Markets, Dominic Y and Geoff and many others for training our squads, and all our schools (not least my own, Trent College) for allowing us the time off to come to Cyprus. [In fact, we always have many schools involved. In our 18 times at the Balkan Olympiad, there have been only two years in which any one school has had more than one of its students in the same UK team; and in the IMO that has only happened for us once this century (in 2010). It was interesting to hear that five of the Romanian team at this competition were from the same school; the same was true for Moldova, while four of the Azerbaijan team were from one school - the UK has never had even half of its team from the same school, in any international maths competition.]

In summary, these events are truly fantastic opportunities, we're all very grateful to have been given the chance to have been involved this year, and our team genuinely regard it to have been one of the most amazing weeks of their lives – long may such things continue!