IMO 2021 - UK Report

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The UK Mathematics Trust[2] organises competitions, mentoring and other enrichment activities for talented and enthusiastic school-aged mathematicians. One strand is a training programme for the country's top young problem-solvers to introduce them to challenging material, enjoyable in their own right, but also the focus of international competitions. Our goal is to use these competitions as a framing device to promote ambitions and excitement for mathematics and a positive scholarly culture amongst all children who attend our events, not just those who end up representing the UK on an international team.

The International Mathematical Olympiad is the original and most prestigious such event, now in its 62nd edition. About a hundred countries are invited to send teams of up to six contestants, and this year I organised the preparation for the UK team to take part in IMO 2021, hosted by St. Petersburg, Russia. As in 2020, the global travel restrictions meant the competition was held in an online format. Each country gathered, where possible, in a single location to attempt the papers in a Covid-secure venue, with light remote invigilation. The UK hosted our exam centre in Leeds, with the exams on July 19th and 20th, and marking continuing remotely via an online platform over the following days.

The whole academic year has been a period of great uncertainty and significant challenges. The UK's usual IMO programme is based around a cycle of residential events, not all of which can pass easily to an online setting. In particular, while the spotlight is often on lectures and sessions, plenty of the learning experience lies in the shadows, or at least in the breaks between sessions, with the chance to meet other motivated young mathematicians, and interact with adult volunteers, who all offer a range of mathematical perspectives. Indeed, even the ritualistic[3] card games that punctuate the day for some of the children serve a valuable purpose to establish a meaningful sense of community. These events are challenging, but also exciting and memorable, and many participants find themselves leaving with renewed motivation to work hard at mathematics.

In conclusion, we very much hope this will be the first and only fully online annual cycle. However, there was still much to appreciate, and in the rest of this report, I will give a brief summary on the UK's results at IMO 2021, followed by a discussion of the problems, and a short, light-hearted diary of our competition week in Leeds.

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[3] Some of the adults who are regularly present may prefer stronger adjectives!

Background and UK results

The UK is fortunate to be able to offer some training events leading to the IMO, and other international competitions. We believe strongly that preparing for such competitions, with some appropriate and sensitive mentoring, is a hugely valuable educational experience.

Our team of six students worked hard to prepare for IMO 2021, and were well-placed for a good performance. The current Year 13 cohort was unusually strong, and the scores on our selection tests, run remotely in April and June, were higher than we expected. A number of students, including our reserve, Tommy Walker Mackay, would certainly have been in the top six in a typical year. A handful of younger students showed great promise too, and we look forward watching their academic growth over the coming years when they will have a good chance to compete with Isaac (Year 10) and Mohit (Year 12) for places in our future IMO teams.

Here are the individual results of the UK team at IMO 2021:

	Q1	Q2	Q3	Q4	Q5	Q6	Σ	
Mohit Hulse	7	0	0	7	7	0	21	Silver Medal
Isaac King	7	0	0	7	7	0	21	Silver Medal
Samuel Liew	7	0	1	7	7	7	29	Gold Medal
Yuka Machino	7	0	1	7	7	7	29	Gold Medal
Daniel Naylor	7	1	1	0	0	0	9	Honourable Mention
Jenni Voon	7	1	0	7	7	0	22	Silver Medal

This represents an excellent team performance, and we are very pleased for them all. At an individual level, this was Yuka's second gold medal at IMO, and the second time she has been the leading female participant, to accompany her three gold medals at the European Girls' Mathematical Olympiad. For three of our team to earn silver medals in their first IMO is also an excellent achievement. As a team, the UK came 9th out of 107 participating countries. When planning our preparation for IMO, we do not typically expect to come in the top ten amidst the tight competition, but this is now the fourth time we have achieved this in six years.

Full results can be found at `https://www.imo-official.org/year_info.aspx?year=2021`.

Commentary on the problems of IMO 2021

These discussions of aspects of the problems are targetting a wide range of readers, and I hope there will be something of interest for everyone. Full solutions to the problems of IMO 2021 are easily found on the internet. Here, we discuss some steps of the solutions, and sometimes some background or tangents.

For up-and-coming students interested in working towards the IMO, I hope that it's clear that trying the problems yourself is also useful. Especially on the more accessible problems, you will probably find the commentaries more interesting after trying to find a solution yourself.

Rather than inject subjective commentary into individual problem discussions, I will say here that I thought these were six good questions, which covered an appropriate range of topics, style and difficulty, and were a good advert for competition mathematics. The papers might have been better balanced if Q2 and Q6 were swapped, but this is a relatively minor point; and one which is unfair to dwell on to excess with the benefit of hindsight.

Problem 1

Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

One simple way to confirm that the result holds is to find $a, b, c \in \{n, n+1, \ldots, 2n\}$ such that the sum of any pair is a square. There are lots of triples of positive integers $\{a, b, c\}$ with this property and, in particular, if we find one such triple, then $\{ak^2, bk^2, ck^2\}$ is also a valid triple, though this infinite collection of triples is too sparse to be useful in this problem for all n (in particular, for the *smaller* values of n).

In practice, we want to construct triples $\{a, b, c\}$ which are as close together as possible, to make it likely that they will fit into the range $[n, 2n]$. One option is to set the squares to be as close as possible, which is achieved by:

$$a + b = (k-1)^2, \quad a + c = k^2, \quad b + c = (k+1)^2. \tag{1}$$

We can solve these simultaneously, but if we want a, b, c to be integers, we need to insist that $k = 2m$ is even. Then $a + b + c = 6m^2 + 1$, leading to

$$a = 6m^2 + 1 - (2m+1)^2 = 2m^2 - 4m, \quad b = 2m^2 + 1, \quad c = 2m^2 + 4m.$$

It's crucial that these can be squeezed into the range $[n, 2n]$. That is, we require

$$2m^2 - 4m \geq n, \quad \text{and} \quad 2m^2 + 4m \leq 2n.$$

It's probably most helpful to reverse the perspective here, and say that a particular value of m covers the case of n precisely when $m^2 + 2m \leq n \leq 2m^2 - 4m$. For example, the case $m = 9$ covers $n = 99, 100, \ldots, 126$. It remains just to check that these intervals for n cover all integers greater than 100, which can be verified by checking that

$$2m^2 - 4m \geq (m+1)^2 + 2(m+1), \quad \text{for all } m \geq 9.$$

Note that this argument doesn't cover the case $n = 98$, which is good, because for $n = 98$ the problem is not true! Indeed, unpacking the construction gives a recipe for a counter-example (since there are not many squares to avoid, and parity[4] can take care of most cases). It's quite cool that a contest problem at the easier end of the spectrum manages to get right down to the true bound, and I think this was a great choice as the first problem for this IMO.

Problem 2

Show that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

This question has been widely criticised, perhaps unfairly. The results indicate that it was found hard, and it's probably reasonable to say it was found too hard. But personally I don't think it's a bad question, and might even have been an excellent question if it had been listed in an harder difficulty slot.

There are two aspects that make it hard:

[4] That is, whether a number is odd or even.

- The principal solution route[5] involves a subtle study of concavity (piecewise, of the function $\sqrt{|\cdot|}$) in order to convert to a form where a very particular inductive step will work.
- This solution route subverts students' expectations of how one might approach an olympiad inequality. Often students would be looking to make insightful substitutions into well-known inequalities, or might be looking for combinatorial reposings of the two sets of sums and differences. Technical 'calculus-lite' methods would generally be a last resort.

I think the second of these is more relevant. It's good to subvert expectations sometimes, and it's true that this type of analysis feels stylistically closer to research. But research problems come with their own context[6], and it's relatively rare that one that one goes into a step like this without some clues about likely routes.

And in this problem, there really did only seem to be one route, namely to aim for an induction under the assumption that there exist $x_i + x_j = 0$. Which is fine, because several students got a mark for noticing that some nice cancellation happens when this holds (and indeed, this is supported by the equality case $(-\lambda, \dots, -\lambda, +\lambda, \dots, +\lambda)$; with equal numbers of each $\pm\lambda$). But there were many possible induction routes, and this is only one that seemed to work. Of course, there might not be such x_i, x_j which sum to zero, but shifting all the values by the same constant changes the RHS but not the LHS, and one can formalise an argument using concavity that transforms the RHS to a case where there do exist $x_i + x_j = 0$, while making it smaller, and enabling the induction step to be completed.

Problem 3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

A lot of IMO geometry problems about triangles are *partially-asymmetric*. A typical situation is that vertex A is special, and vertices $\{B, C\}$ play symmetric roles. Personally, I normally draw such a triangle with A at the top, and BC as a horizontal base, unless it becomes clear there's a better way to visualise it. Sometimes a triangle configuration is completely symmetric, and the roles of all three vertices $\{A, B, C\}$ are interchangeable, though these are less popular.

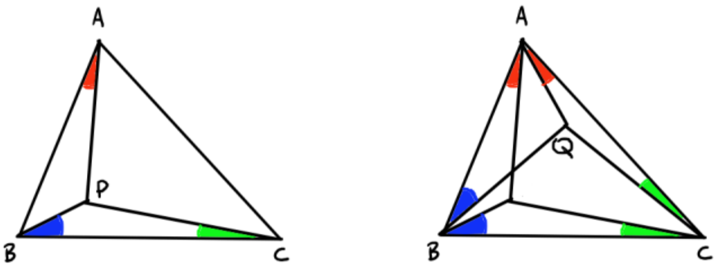
This configuration is initially symmetric in $\{B, C\}$, but then the introduction of X breaks this symmetry, and this brings lots of challenges in finding routes into the problem. I've read the solutions and found them all quite hard and technical, so I'll just talk about the beginning.

For some students, the first step was to establish that $BCEF$ is cyclic. This is something one might conjecture. Ultimately, this would give a bit more control over the intersection of BC and EF, eg via Power of a Point and other related methods such as inversion. I conjectured this, checked it using some diagram software, but made no progress initially. There doesn't seem to be a direct argument for this, but the key step is to analyse the angle condition, and try and distill this into a more useful form. Sometimes we try and achieve this via reflections or translations. But because of the symmetry in $\{B, C\}$, and the location of the equal angles, it's particularly useful to consider *isogonal conjugacy*. Let's define this.

[5] The problem selection committee mention a couple of solutions involving *bilinear forms* but it seems unlikely that schoolchildren would go looking for this, even if they knew some of the underlying theory.

[6] And, except in unusual cases, without short-term time-pressure.

Consider a triangle ABC and a point P. For simplicity, we'll assume that P is inside $\triangle ABC$. Then if we reflect AP in the angle bisector of $\angle A$, and similarly for BP, CP in the angle bisectors of $\angle B, \angle C$, we get three new lines, and the key result is that they meet at a point Q. This procedure is clearly reversible, and we say such pairs of points (P, Q) are *isogonal conjugates* with respect to $\triangle ABC$.



A key example is (O, H), the circumcentre and the orthocentre. The incentre I is its own isogonal conjugate and, relevantly here, when D is a point on the $\angle A$-angle bisector, its isogonal conjugate D' also lies on this angle bisector. If you haven't explored the configuration for this problem in detail yet, you might like to introduce this isogonal conjugate D' to your diagram and see whether you can spot any useful features that emerge. You might then be able to prove that $BCEF$ is cyclic, which was one of the initial points of progress in the full solution to this hard problem.

Problem 4

Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC. \tag{2}$$

Geometry questions at the IMO often follow a pattern of taking a relatively simple configuration, and finding a way to disguise the most helpful interpretation. This problem was slightly different, as essentially everything you need to solve it is given in the statement. It's also kind of remarkable that the conclusion can be framed entirely in terms of two equal perimeters (of two quadrilaterals). While it might get tiresome to have a problem like this every year, I thought this was a novel and attractive variation.

The conclusion involves equal sums of lengths, and we know (by applying equal tangents at each of A, B, C, D) that[7] $AB + CD = AD + BC$, so we will bear this in mind. In (2), the most awkward pair of lengths might be TX and YZ, since neither of these lines are tangent to Γ. However, one might conjecture from an accurate diagram that they are equal, and because they are both chords of circle Ω, this gives a recipe for proving it.

In particular, though it's not emphasised as much as circle theorems, it's worth remembering the characterisation of *isosceles trapezia* that any two of the three conditions i) cyclic; ii) $TX = YZ$; iii) $TZ || XY$ imply the third. In this case, to establish that $TXYZ$ is an isosceles trapezium, it's probably easiest to use angles more directly. Since AI is the (external) angle bisector of

[7]This is also known as Pitot's theorem.

$\angle XAY$, we get[8] $IX = IY$, and arguing similarly in $\triangle CTZ$, one gets $IT = IZ$ too, and so it follows that $TXYZ$ is an isosceles trapezium. We will make a further observation: the axis of symmetry of this isosceles trapezium is the diameter through I.

We've now handled lengths $TX = YZ$. Some students argued for the remaining lengths with some similar triangles or calculations involving the sine rule, but in fact it's enough to use equal tangents. If you break the length AD into two pieces, we can reassemble them to show that $AD + DT + XA$ is equal to the sum of lengths of the tangents from X and T to Γ. And now the stronger observation we made about symmetry for the trapezium $TXYZ$ is really valuable. Because I is the centre of Γ, and so the sum of the lengths of the tangents from X and T to Γ is equal to the sum of the lengths of the tangents from Y and Z to Γ!

Problem 5

Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Replacing 2021 by n, observe that the problem is false if n is even. One could colour the walnuts $1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ red, and the other half blue, then choose any initial ordering which alternates red and blue nuts. From this, we conclude we'll need to find some global property of the ordering(s) to study.

We will probably be assuming for contradiction that we always have either $a, b < k$ or $k < a, b$. One of my initial ideas was to study the number of times the permutation goes *Up-Down* or *Down-Up*, because, under this assumption, this is preserved on either side of walnut k. However, it is not always preserved globally, eg in $(1, 6, 7, 4, \mathbf{5}, 2, 3) \mapsto (1, 6, 7, 2, \mathbf{5}, 4, 3)$.

The issue in this example is that we are over-specifying the roles of 2 and 4. We really just want acknowledge that they are both smaller than **5** (or, in other cases, larger). In the language of the question, this means that they have both already been used to generate a swap. Counting the number of such nuts which have already been involved in a swap is boring, because it's a sequence that goes $0, 1, 2, \dots, n$ as the process evolves. However, we're actually very close because counting *pairs of adjacent* such nuts is interesting, because this changes by two (either up or down) in any move.

When writing up, we avoid the pre-amble, and jump straight to the most useful classification. But it's worth noting how the 'magical' step was actually pretty close to an experimental and unsuccessful step. So we track how many pairs of walnuts with values $\leq k$ are adjacent after the kth move. This is initially zero, and is equal to n at the end. But since it changes by two in any move, this means n must be even, and we get a contradiction in the relevant case n odd.

[8] Several years ago, during a discussion of the merits of learning long lists of theory, I frivolously described the internal version of this result as "Dominic's favourite fact, number 51", and this name proved memorable to some of the students. The internal version states that the angle bisector of $\angle A$ meets the perpendicular bisector of BC on the circumcircle of $\triangle ABC$, and so in fact at the 'arc-midpoint' I_M of arc BC. Adding the incentre I and A-excentre I_A, we also have $BM = CM = IM = I_AM$. Some people call various versions of this result the *trillium theorem*, and some consider it 'well-known', which is probably reasonable in contexts such as the IMO.

This problem is really about the process rather than about any of the individual permutations, but the general topic of *pattern-avoiding permutations* is interesting. For example, one might ask how many permutations $(a_1, \dots, a_n)$ of $\{1, 2, \dots, n\}$ have no three indices $i_1 < i_2 < i_3$ such that $a_1 < a_2 < a_3$? This is going to force the permutation to be the union of two decreasing subsequences[9]. However, avoiding more complicated patterns can rapidly become quite subtle, for example no four indices $i_1 < i_2 < i_3 < i_4$ such that $a_2 < a_1 < a_4 < a_3$, and enumerating and generating (eg uniformly or almost-uniformly at random) such permutations remains an active topic of research[10].

Problem 6

Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and $B_1, B_2, ..., B_m$ subsets of A. Suppose that, for every $k = 1, 2, ..., m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements.

The solution here is perhaps deceptively simple. One could focus on A: how to choose its elements? how to assign the elements to the B_ks? But instead our argument takes the form: "if we can do X using $\mathcal{X}$, then we can do Y using $\mathcal{Y}$" as follows.

We know how to write any positive integer in terms of sums powers of m. We regularly do this in base 10! However, if we aren't allowed to use any non-zero unit digits, then we can only write multiples of 10 in this form. A similar effect applies here. For every $k \in [0, m^m - 1]$, we can write

$$mk = \beta_1 m + \beta_2 m^2 + \ldots + \beta_m m^m, \quad \text{for } \beta_i \in \{0, 1, \dots, m-1\}.$$

However, we can also each of these powers of m as a sum of elements of A, which we can represent as

$$m^i = \gamma_1^{(i)} a_1 + \ldots + \gamma_n^{(i)} a_n, \quad \text{for } a_i \in \{0, 1\}.$$

Combining these, we obtain:

$$mk = \left(\beta_1\gamma_1^{(1)} + \beta_2\gamma_1^{(2)} + \ldots + \beta_m\gamma_1^{(m)}\right) a_1 + \left(\beta_1\gamma_2^{(1)} + \cdots\right) a_2 + \ldots + \left(\beta_1\gamma_n^{(1)} + \ldots + \beta_m\gamma_n^{(m)}\right) a_n. \tag{3}$$

This notation is cumbersome and annoying. But, importantly, each of the bracketed terms is between 0 and $m(m-1)$. In particular, since the a_is are fixed, the RHS can take at most $(1 + m(m-1))^n$ values. If $n < \frac{m}{2}$, then this is a contradiction because we've set up the LHS mk to take m^m values.

Having solved the IMO problem, it's good to ask how tight this bound is when m is large. Just for ease of description, it's convenient to describe $(\beta_1, \dots, \beta_m)$ as a vector $\mathbf{b} \in [0, m-1]^m$, and similarly $\mathbf{a} = (a_1, \dots, a_n)$, and $\mathbf{m} = (m, m^2, \dots, m^m)$. Finally, we need $\Gamma = (\gamma_j^{(i)})_{i,j}$ a matrix in $\{0, 1\}^{m \times n}$. Then the statements above say $mk = \mathbf{b} \cdot \mathbf{m}$, and $\mathbf{m} = \Gamma\mathbf{a}$, leading to $mk = \mathbf{b}\Gamma\mathbf{a}$.

For the argument above to be tight, we would require that as $\mathbf{b}$ varies in $[0, m-1]^m$, the quantity $\mathbf{b}\Gamma\mathbf{a}$ takes on the order of m^m values in $[0, m^{m+1}]$. As often in this type of situation, it's helpful to think of this quantity in the opposite direction to how we derived it, ie as $(\mathbf{b}\Gamma) \cdot \mathbf{a}$. But a given component $[\mathbf{b}\Gamma]_i$ just consists of a sum of some of the β_is (since each entry of Γ is 0 or 1), and there will be considerable overlap.

[9] Some readers may find it interesting to try and count these either directly, or by comparison with other combinatorial families you know how to enumerate.

[10] Ambitious readers trying to follow up on this may find the introduction of Baxter + Pudwell *Enumeration schemes for vincular patterns* (2012) to be interesting and readable.

To quantify the overlap, it's helpful to let $\mathbf{B}$ be a uniformly-random choice from $[0, m-1]^m$. Then $[\mathbf{B}\Gamma]_i$ is an IID sum of a $k(i)$ copies of a uniform choice from $[0, m-1]$, where $k(i)$ is the number of $\gamma_j^{(i)}$s which are 1. The variance of a single such uniform choice is $O(m^2)$, and so the variance of such a sum is $O(m^3)$. So if we restrict to, for example, 99% of the possible $\mathbf{b}$s, then $[\mathbf{b}\Gamma]_i$ takes $O(m^{3/2})$ values which is, note, a non-trivial reduction on our initial bound of $O(m^2)$.

Because we have m coordinates, we need stronger bounds in probability to take a union. In particular, one can argue using a Chernoff bound[11] to obtain a *moderate deviations* estimate[12] that the probability that $[\mathbf{B}\Gamma]_i$ lies outside a particular interval of width $m^{3/2+\epsilon}$ decays exponentially in m. In fact, we'll say that it decays $\sim e^{-m^{\alpha(\epsilon)}}$, and be vague here about what $\sim$ means, and how $\alpha(\epsilon)$ decays with ϵ.

This then means that the probability that any coordinate of $\mathbf{B}\Gamma$ lies outside its interval is at most $\sim me^{-m^{\alpha(\epsilon)}}$, by a union bound. (At least, for large m.) So, returning to the form of the original argument, where we compared the number of ways to write each side of (3), we have

$$\left(m^{3/2+\epsilon}\right)^n + m^m \cdot me^{-m^{\alpha(\epsilon)}} \geq m^m.$$

So for large m, we must have $n \geq (\frac{2}{3} - \epsilon')m$, where $\epsilon' \to 0$ as $\epsilon \to 0$.

I suspect an asymptotic bound of $n \gtrsim \frac{2m}{3}$ is also not tight, but further refinement would probably require more detailed number theory analysis.

[11] A Chernoff bound refers to the notion of obtaining strong upper bounds in probability on the event $\{X > z\}$ by applying Markov's inequality to the quantity e^{tX}, obtaining $\mathbb{P}(X > z) \leq \frac{\mathbb{E}[e^{tX}]}{e^{tz}}$, for every choice of t. In this case, the random variables are bounded, in fact between 0 and $k(i)m$. After centering around their expectation, a good trick is to approximate the uniform distribution U_K on $\{-K, -(K-1), \dots, 0, \dots, K\}$ by the uniform distribution $\tilde{U}_K$ on $\{-K, K\}$. There is a notion of *convex ordering* under which $U_K \leq_{\mathrm{cv}} \tilde{U}_K$, and in particular, we have $\mathbb{E}\left[e^{tU_K}\right] \leq \mathbb{E}\left[e^{t\tilde{U}_K}\right]$ by Jensen's inequality. Note that calculating this moment generating function is easier for $\tilde{U}_K$ than for U_K. This argument is essentially the same as that for the *Azuma–Hoeffding* inequality, which is stated in terms of martingales with a.s. bounded increments, but could be applied directly in this setting.

[12] The non-vanishing (in probability) deviations in the previous paragraph lie in the domain of the Central Limit Theorem. In this context, a *large deviation* involves a multiplicative factor between the observed sum and its expectation as $m \to \infty$. See *Large Deviations Techniques and Applications* (Dembo, Zeitouni, 2009) for a treatment in high generality of such phenomena. Petrov (1976) is a good starting point for exact asymptotics for moderate deviations of random walks. Such non-asymptotic *concentration estimates* are often studied as part of a first course on *Mathematical Machine Learning*, and can also be found in many classical texts in probability.

Brief IMO diary

Saturday 17th July

It's one of the warmest days of the year, and the journey from Oxford to Leeds makes me long for car ownership. One of our deputy leaders, Emily, has been sidelined at the last minute by an ominous message from the NHS track-and-trace app, but it's great that Tom has been able to step in at short notice. It's easy to forget that, though a long weekend in the Leeds Park Plaza is less exciting than some IMO trips of recent memory involving a fortnight in, for example, Thailand, for the students especially it's a great deal more interesting than, for example, January-March 2021! Isaac could have three more IMOs in which to see the full experience, but we admire the others for embracing the positives of this unusual edition.

Although we are not in Thailand, they get to experiment with various chopstick challenges while we dine at the hotel's restaurant. It claims to be one of the top 10 pan-Asian restaurants in Leeds, which is likely to be true, though we were their only customers tonight, and the portion sizes for dessert lean towards generosity, challenging anyone's post-lockdown diet plans.

Sunday 18th July

The morning we run a final practice exam for the students. This is a chance to figure out how to turn the webcams on and the air conditioning off. As per tradition, this final warm-up paper is shared with the Australian team, the marks are added up, and the winner is declared to have earned the *Mathematical Ashes*. Some spirit is lost by starting 12 hours after their rivals, but it is still a valuable experience. The opening ceremony is live-streamed from Russia during lunch, and Mohit, currently stuck in Bangalore, has organised a live watch-party with the Leeds contingent. They have all chosen novelty backgrounds for their introductory cameos during the ceremony, though Jenni's rainbow queens[13] have survived the crop better than the others.

Deputy leaders Sam and Tom get started on the marking after lunch while I sort out the logistics. As it becomes clear that the work is considerably more complicated to mark than we'd anticipated, I feel like I'm getting a generous deal. Then the full horror of the rental printer situation dawns. The IMO papers tomorrow and Tuesday will each involve scanning up to 150 pages, and it seems wise to make sure we are familiar with the equipment. There are repeated calls for help, and at one point it is suggested to "maybe reinstall the drivers each time you have a new document to scan?" Beyond this we draw a veil of discretion, since it did in fact work adequately when required, though I no longer felt guilt about assignment of duties.

Meanwhile, the students have been bowling. Apparently Jenni was the champion in both rounds, but more discussion is devoted to the nature of the alley, and whether the eventual decision to leave one barrier up should be considered as an example of a 'compromise'.

Monday 19th July

Today was the UK's supposed '*Freedom Day*', and it's true that at both 6am and 10pm, the amount of leopard print on display in Central Leeds was striking.

More relevantly to us, it is Day One of the IMO and, along with Ukraine, Uganda, plus a few other countries in the online Zoom invigilation room number 19, our 4.5 hours starts at 8.30am. As discussed earlier, Questions 2 and 3 are found hard by the UK students. This has happened at in-person competitions in the past, and often the post-exam atmosphere is bleak. Somehow

[13] `https://youtu.be/DpZnQuUI27Y?t=4295`

on this occasion we can just ‘feel’ that they are supposed to be hard, despite none of the adults or children knowing a solution, and people are happy enough to put on an upbeat mood. To have a full set of solutions to Question 1 is very pleasing, as this seems like a potential banana skin, even for experienced students.

We have arranged to take the students for crazy golf nearby, as a late-afternoon distraction from post-exam analysis and ennui. Hannah, the Executive Director of UKMT, joins in with great competitive zeal. The aesthetic of our chosen golf course is borrowed from Hammer Films, and Samuel and Daniel take a number of photos of the team which, on reflection, may not be suitable for use on the front cover of the UKMT yearbook.

Tuesday 20th July

Day Two of the IMO can often feel strange, as the nerves have partly diminished now that everyone knows they’ve solved a question, and figured out how the webcams and the page-numbering rubrics work. But we’re also conscious that it’s the last day of an activity which, for our graduating students especially, has been a big part of their lives for several years. In any case, it’s a shame for Daniel that today’s questions didn’t align as he would have hoped, but it doesn’t diminish his mathematical progress and achievements over the past year. For the other three British students in Leeds, the paper seems to have gone very well, especially for Samuel, who seems confident of solving all the problems; even more so, after our post-exam Zoom with Yuka in Tokyo, who describes very similar approaches to each of the questions.

Time passes quickly, and it is less than two years since Jenni and Daniel attended their first IMO camp, in Oxford where I gave a series of lectures on geometry. They have printed a mug quoting one frivolous tangent[14] from this masterclass as a leaving present, which is a really lovely gesture, and I have already drunk several mugs of (for now, iced) tea from it with appreciation[15]. In any case, this draws to a close the in-person part of their IMO journeys, and I’m delighted that the group is leaving on a high.

Sam and I linger to address the work. Markschemes and solutions have appeared so, after a brief pause to reflect that we don’t feel bad for not solving Q2 or Q3, we get to work and, with remote help from Tom and Emily, we are basically finished with the academic components of IMO 2021 in time for a curry.

Wednesday 21st and Thursday 22nd July

Traditionally, after the IMO, while the leaders wrangle for marks with the host country’s co-ordinators, the students are taken on some excursions, which are sometimes memorable for good reasons, and sometimes memorable for other reasons. Sadly, these excursions are another casualty of the online format. The wrangling is also a casualty, but maybe there’s cause for reflection on how much more smoothly and non-acrimoniously the marking is finalised when managed through an impersonal portal.

I’m taking my own excursion round the Yorkshire’s finest landscapes. This was the apex of the heatwave and the Dales do not have any trees, or other forms of shade. It does, however, have strong enough signal from the tops of the Yorkshire Three Peaks that it is easy enough to complete the online coordination process in transit. The staff are particularly pleased that we’ve scored 7 for every script in which our students found a solution. No marks were dropped anyway for carelessness or obscurity, and this is worth celebrating!

[14]See earlier footnote to Q4

[15]I’m also grateful for the t-shirt, though I can’t promise that this will ever be worn in a professional setting...

Despite the lack of travel opportunities, the students have enjoyed some online talks from notable contemporary mathematicians including László Lovász, Stanislav Smirnov and Lisa Sauermann. We also find a video of our own Geoff Smith firing the noon-day cannon at the harbour in St. Petersburg. Given the global student reaction to the papers, I hope the author of Problem 2 had reliable access to an underground bunker.

Friday 23rd July

It is unfortunately the nature of the beast with online competitions, that the programme can drift away, without the focus of activities and interaction in a single location. I'm now back in Oxford, and all our students are home too. There is a second, and final meeting of the IMO jury, consisting of all the leaders, on Zoom. There are some short discussions, and it will be interesting to see whether some of the innovations necessitated by the online format are retained when we are able to meet again in person, hopefully in Oslo for IMO 2022.

This meeting also includes the revelation of the medal boundaries. As always it is hard to guess how other nations will have found the problems, but it becomes clear that others have struggled as much as we did on Problems 2 and 3, and consequently the scores are low. Samuel and Yuka are delighted with their gold medals, as they should be, and everyone is pleased with the team effort which has earned UK another year in the IMO top ten.

Thoughts drift back to moving home, applying for jobs and, for our Year 13 students, preparing to start the next chapter at university in the autumn. Daniel, Jenni and Samuel will be heading off to Cambridge soon, while Yuka is preparing to move to Boston where she will be studying at MIT. We hope their experience with these olympiads will be an useful springboard for creativity, enjoyment and success in the next stages of their mathematical journeys.

Conclusion

Taking a UK team to a virtual IMO often requires just as much effort, if not more, than an in-person IMO. Thanks are particularly due to:

- Kit Richardson, and the other staff at UKMT, for help in organising and overseeing all the events, especially the IMO venue in Leeds.
- Our Australian colleagues, Angelo and Sampson, with whom we shared training material even if not, unfortunately, a training camp this year. Also Ismael Sierra, who was extremely helpful and competent as our IMO commissioner, ensuring smooth progress between our site in Leeds, and the Russian organisers.
- The organisers of IMO 2021, including our own Geoff Smith, president of the IMO board, who did a great job in 2020 of ensuring the competition went ahead at all; and developed the schedule this year to create an engaging and sustainable model for online IMOs.
- All the mathematicians, young and old, who freely offered sessions at our online training events, and helped mark our selection tests and practice papers. We all very much hope it will be possible to reconnect in person as a community during the next cycle.
- In particular, I'd like to acknowledge our Deputy Leaders Sam Bealing and Emily Beatty, who helped organise a superb range of sessions, entertainment, and mathematical material for the students over recent months, even in the middle of internships and their final university exams. We also thank Tom Hillman who stepped in at minimal notice, and brought great enthusiasm and expertise to the contest days and the marking.

- Finally, of course, our UK team comprising Mohit, Isaac, Samuel, Yuka, Daniel, and Jenni. Despite all the challenges of the past academic year, they embraced the opportunities to push themselves mathematically, and learn from their leaders and from each other. Their success at IMO 2021 was hugely deserved.