Balkan Mathematical Olympiad 2019 – Student Report

The UNKs

30th April – 5th May 2019

Day 1 – Tuesday 30th April 2019

In keeping with tradition, the UNKs gather at Stansted Airport outside WHSmiths, although we soon retreat to Costa due to the existence of tables which aid with Dominic's welcome in the form of paper-folding exercises.

Problem -6 (Dominic)**.** *Take a piece of paper and mark a point P on one of the sides. Create two folds taking P to the two corners of the side opposite that containing P. Let the two creases this creates intersect at X. What can you say about X?*

Problem -5 (Dominic)**.** *Take a piece of paper and mark a point P somewhere in the interior. Make multiple folds taking P to various points on one of the sides. The collection of resulting creases will eventually form a curve. Describe this curve.*

Problem -4 (Dominic)**.** *Take a circle with centre O and mark a point P distinct from O in the interior of the circle. Make multiple folds taking P to various points on the circumference of the circle. The collection of resulting creases will eventually form a curve. Describe this curve. What happens when P is outside the circle?*

Being quiet, well-behaved and non-trouble-causing mathematicians, we all arrive before the target of 12:30. We are informed that Geoff will not be joining us and that Dominic Yeo will fly out on Thursday to aid with marking 'in case we try strange non-synthetic ways of doing geometry'. We pledge to make him useful.

Having successfully remembered to discard their water bottles, and packed contraband such as compasses and scissors in their checked luggage, the team clear security without a hitch and head for lunch where we discuss among other things the shape of the water stream from a tap. Despite using our knowledge of physics, we do not manage to explain the shape mathematically and instead sidetrack our discussion with the potential counterexample of aerated taps. However the puzzles don't stop, but instead switch to the topic of countable and uncountable sets:

Problem -3 (Dominic)**.** *Can we have uncountably many non-intersecting circles in the plane?*

Problem -2 (Dominic)**.** *Can we have uncountably many non-intersecting infinity signs in the plane?*

Problem -1 (Dominic)**.** *Suppose we have a family of subsets of the natural numbers such that for every two subsets in this family, one is a subset of the other. Must this family be countable?*

These problems create ample discussion for the remainder of the boarding process, with problem -1 remaining elusive for the rest of the trip. The plane cabin erupts with energetic applause as the wheels make their first contact with Moldovan soil, despite the pessimists among us being fully aware that the plane is still moving at some 150mph. On the other side of passport control we are greeted by several Moldovan students, including our guide Valeriu who will look after us during our stay. We are asked whether we like sports, which is perhaps an unusual question to ask a group of maths students. We are handed Moldovan apples, and as we wait for our luggage to appear on the belt, we decide to pose with our new-found mascots.

After quickly eating dinner at the host school, we make our way to the hotel where all the contestants will be staying for the duration of the contest. We soon discover that due to the parity of the number of male and female team members, Patrick and Alevtina would be sharing rooms with contestants from other teams with similar parity issues (Saudi Arabia and Italy respectively). However the rest of us were not in the clear yet, as it soon emerged that the two fully British occupied rooms had only one double bed each. Liam and George took an active approach in resolving the situation and after alerting Ava, were quickly relocated to a room with correct sleeping arrangements. Brian and Thomas, not wanting to cause a fuss, simply lived with the inconvenience. Such is the price of inaction.

The UK's mascots did not survive for long after this picture.

Day 2 – Wednesday 1st May 2019

We have an early 7:30 start for breakfast at the school. Our guide gives us a walking tour of the school which has an impressive range of facilities including a swimming pool, a room dedicated to guitars and a room containing several thousand pounds worth of robotised LEGO. We return to the hotel to change for the opening ceremony. Our polo-shirts are the same colour as last year "because they were cheap". We don't mind as there have been much worse colour choices in the past.

We have some spare time so we decide to explore the town a little. Moldova mainly follows Orthodox Christianity, so their Easter took place the previous weekend, and decorations are still up in the nearby park. We pose in front of a giant egg before returning via the aptly named 'Str. 31st August 1989'.

Before the ceremony, local TV stations patrol the aisles hoping to extract some words from the contestants. George and Brian are the only ones willing to sacrifice their dignity for the chance to appear on Moldovan television.
The ceremony itself adheres strictly to the 1 hour it has been allotted, and fills as much as possible of it with traditional dancing and singing. A theme of patriotism is evident, with dances chosen to 'demonstrate the agility of the Moldovan people'. We note that every act is followed by two rounds of applause – the first is as standard and the second after the enthusiastic hosts tell us to 'thank the beautiful dancers' (we conjecture this is scripted in case the first applause does not exist for whatever reason). Each team stands up in turn with their flag,

naturally leading to the following problem:

Problem 0 (The UNKs)**.** *Which way up does the Union Jack go?*

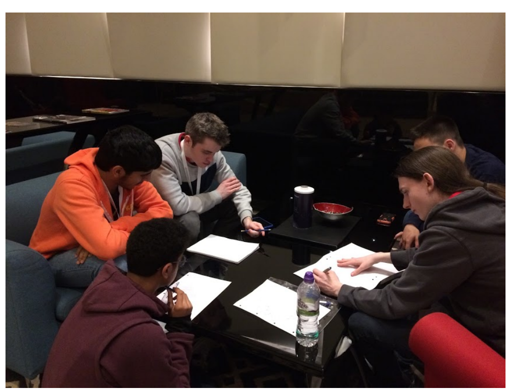
After lunch at the school, we are given an opportunity to visit the contest room and find out where our seats will be. We are not sure why this information will be particularly useful but we decide to go nonetheless. The UNKs then split up to do the sports of their choice. George, Alevtina and Patrick (being the only ones to have brought swimming gear) head for the pool while the rest try their hand at ping-pong.

The youngest stars of the opening ceremony

The swimming teachers at the school decide to try to organise some relay races which of course requires the separation of the swimmers into two non-intersecting subsets with everyone included in their union. George and Patrick are arbitrarily made team captains, which puts them in the awkward position of having to gauge a person's swimming ability purely from their body composition. We then discover that the races have the quirk that arm movement is forbidden, something that was probably lost in translation. The teachers seem to be fond of shouting, but we are never sure whether they are telling us what we are doing wrong, telling others what they are doing wrong or simply encouraging us. We are never quite sure what we are doing but one Bosnian student appears to be impressed with his captain's performance, which will prove to be the only time the UK gets praise from another country for sporting ability.

After dinner (which is brought forward by an hour), we hang around in the lobby and briefly collaborate with members of the Saudi team on a geometry problem. We then decide we are bored of maths and play cards instead. Ava kindly offers to buy us drinks (non-alcoholic of course – that can come later) from the bar. After a good hour or so, the management interrupt us to inform us that card games are not allowed in the lobby due to the existence of a casino on floor 2. We find this rather strange since Irish Snap doesn't tend to be played in casinos, not all of us are of legal gambling age and we are paying customers of an otherwise deserted bar. We decide in future we will meet in one of the two UNK-only bedrooms for cards, which is complicated by the logistical issue of the fact that in order to access a floor $n > 2$, we must have a room on floor n. This requires us to all meet in the lobby before being "escorted" to the room by its nighttime inhabitants. Such are the struggles of staying in a 5* hotel. . . .

We decide to play a game that goes by many names, most of which probably wouldn't be allowed to be published here. We eventually decide on the name *'Fecal Face'*. If you know, you know.



A rare example of the UNKs actually doing maths rather than merely talking about it

Day 3 – Thursday 2nd May 2019

The day of the exam. If the 7.30 start yesterday wasn't bad enough, they managed to shave 15 minutes off that today. What's worse, we would be waiting around in the exam hall for 30 minutes from 8.30 to 9.00, after which the exam officially began. Some candidates opted to walk around the exam room to wish their compatriots good luck. The UNKs on the other hand stuck to the GCSE protocol they had been drilled with and remained seated. At around 8:45, the instructions are read out, and we separate our paper out (which thankfully does not involve coloured carbon paper). This does not take long, so we have a silent, tense final five minutes sitting on the grid before the flag is dropped at precisely 9:00 AM and the one hundred and seven envelopes are frantically opened.

Problem 1. *Let $\mathbb{P}$ be the set of all prime numbers. Find all functions $f : \mathbb{P} \to \mathbb{P}$ such that*

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$$

holds for all $p, q \in \mathbb{P}$.

Problem 2. *Let a, b, c be real numbers, such that $0 \le a \le b \le c$ and $a + b + c = ab + bc + ca > 0$. Prove that $\sqrt{bc}(a + 1) \ge 2$. Find all triples (a, b, c) for which equality holds.*

Problem 3. *Let ABC be an acute scalene triangle. Let X and Y be two distinct interior points of the segment BC such that $\angle CAX = \angle YAB$. Suppose that:*

1) *K and S are the feet of perpendiculars from B to the lines AX and AY respectively.*

2) *T and L are the feet of the perpendiculars from C to the lines AX and AY respectively.*

Prove that KL and ST intersect on the line BC.

Problem 4. *A grid consists of all points of the form (m, n) where m and n are integers with $|m| \le 2019$, $|n| \le 2019$ and $|m| + |n| < 4038$. We call the points (m, n) of the grid with either $|m| = 2019$ or $|n| = 2019$ the boundary points. The four lines $x = \pm 2019$ and $y = \pm 2019$ are called boundary lines. Two points on the grid are called neighbours if the distance between them is equal to 1. Anna and Bob play a game on this grid. Anna starts with a token at the point $(0, 0)$. They take turns, with Bob playing first.*

1) *On each of his turns, Bob deletes at most two boundary points on each boundary line.*

2) *On each of her turns, Anna makes exactly three steps, where a step consists of moving her token from its current point to any neighbouring point which has not been deleted.*

As soon as Anna places her token on some boundary point which has not been deleted, the game is over and Anna wins. Does Anna have a winning strategy?

There were two schools of thought concerning post-exam discussion. One approach was to give minimal discussion of the paper beyond problems solved and a high level overview of the solution. The other is to launch a fully in depth discussion, in an attempt to judge the validity of one's solution from the reassuring nods of one's peers while in the lunch queue.

We are reunited with the two Dominics and we discuss our thoughts on the paper. Everyone claims at least one full solution, with most claiming two. Dominic R says he did not think the paper suited the UK skillset and was pleased we all managed to get something out of it. We leave the school and visit the local art gallery while the Dominics pore over our scripts. It seemed such large groups were a rarity at this gallery, as the staff waived the usual policy and allowed us to take a group photo.

We are back at the hotel quite early, so we retire to our air-conditioned XOR window-ventilated rooms.[1] Alevtina discovers she is locked out, and the hotel staff don't seem to be in a rush to fix this. At least she is spared from the elevator (which is playing a loop of 'Everybody Dance Now' on a TV screen) for a few minutes longer. The rest of us are just glad that the advice to bring a spare toilet roll in the ever-comprehensive guidance sent by UKMT was not needed.

After more rounds of cards, we head to a nearby restaurant serving the local cuisine. The waitress advised us to order our mains before the starters arrived to avoid a long wait. Some of us were unsure how much the starters would fill us up, but we were assured that they would only be small. This looked like it would be the case until two plates containing assorted Moldovan pies emerged, which proved to be both extremely popular and filling. Despite this, we manage to finish an impressive proportion of the food ordered.

Credit has to be given to the menu design, which had three languages (including English) on every page. We were initially confused by the fact there were two numbers by every dish, but we quickly learned that one was the price and one was the weight. Some dishes even had breakdowns of the weight, so you would know you were getting precisely 150 grams of chocolate cake served with 60 grams of ice-cream. As mathematicians we can only hope this practice spreads.

Once back in the hotel, we are told to meet for 7:20, which we decide is significantly later than yesterday's 7:15 start and so grounds for more cards and a later night.

Day 4 – Friday 3rd May 2019

As promised, we meet early-but-not-early and make our way to the school. However, for reasons unknown to us we are made to wait and do not actually get breakfast until nearly 8am. There is then another long wait until we leave the school on foot to walk some 10 minutes to the coaches that will take us on our excursion to a monastery around an hour from the capital. The safety and security of a party of young mathematicians is taken very seriously in Moldova, and a police car heads the convoy.

After we return and have lunch, we decide to play basketball with the Turkish team. Fortunately the guides mix up the team, but before long the UNKs on both teams discover that the winning strategy appears to be passing to a Turk then standing back and letting them do their thing. Once the first game is over we decide to retreat to the ping-pong room.

[1] when the latter is opened, the former turns itself off, and we can't decide which is more effective

We meet the two Dominics outside, and they tell us our scores have been mostly finalised. Instead of talking to each of us separately, he simply holds up a page in his notebook with a table in it. For more detail on the process of coordination see Dominic R's report.

However, one score remains undecided. Alas, Patrick's score on P1 will be determined by the answer to the following problem:

Problem 5 (The Coordinators)**.** *Suppose a named theorem exists that allows a contestant to immediately handle the p=2 and p=3 cases for Problem 1. Should the candidate be deducted 2 marks for using this theorem? You may assume the proof of this theorem is not within reach of the contestant.*

The UK team are unqualified to answer it, although the Dominics believe the answer is "no". We turn our attention to a related question:

Problem 6 (Literally Everyone)**.** *How do you pronounce Mihăilescu?*

We head back to the hotel to rest before emerging to go to a pizza restaurant. The beer order is mixed up, much to Dominic Y and Brian's delight as it gives them an excuse to drink the surplus. Only George is sufficiently daring to order his pizza in "XL" size. It arrives on a large wooden turntable for ease of sharing, but since George is not using it for its intended purpose it effectively becomes an extremely unstable plate, meaning Alevtina is recruited to steady it during the industrial-scale dissection.

George and his XL pizza

George having finished his XL pizza

The Dominics leave before dessert in order to attend the final jury meeting where Problem 5 will hopefully be solved. Unfortunately this was in vain as late coordination pushes the meeting back by an hour and a half. The rest of us ponder the menu which advertises a "Desert Day And Night". Some of us conjecture that because deserts are hot in the daytime and cold in the night-time, the dish will incorporate both hot and cold components. We ask the waitress and we are told it is a white and dark chocolate dessert. Several of us are tempted, but when we come to order we are told it is out of stock. Thankfully the chocolate fondant cake proves to be a more than adequate substitute which also fits our conjectured description.

We go up to a room to play cards, and at around 10:30, we are informed that the infamous Problem 5 has been proven false by democracy. This is good news for Patrick's mark, and he celebrates by declaring that if the bronze boundary is 15 he will self-defenestrate. Thankfully when the boundaries arrive an hour later he does not keep his promise. We meet the Dominics in the lobby when they return at around midnight where they congratulate us on five medals before the team quickly retreat to their beds.

Day 5 – Saturday 4th May 2019

After yesterday's wait before breakfast, we elect to meet at 7:40 in order to make more efficient use of our time through sleeping. This turns out to be well-judged and we are shown straight to the dining hall. As we leave it, we see a large group of young mathematicians crowding around a small portion of the wall. Upon closer inspection, this turns out to be the final ranking of contestants. Despite already knowing our results and medals, we feel the urge to join the huddle, to look for interesting statistics if nothing else. Patrick reasons that since his Honourable Mention is rarer than the bronze medals of the rest of the team (and indeed the silver medals too), it is the superior award.

Our second excursion takes us to Orheiul Vechi, an ancient site containing monasteries both above and below ground. The weather is beautiful, as is the view from the top of the hill. We pause for some pictures at the top of a large cliff. Brian makes the bold conjecture that if someone were to fall off the edge, they would not suffer serious injuries. We resist the urge to test that hypothesis. At another more lethal cliff, George joked that were Brian to fall, there would be one fewer person on the IMO squad. Both sides laughed, but the message was sent.

During the return walk from the church at the top of the ridge, the UK team decide to separate from the rest of the group in order to see the underground monastery. We feel oddly rebellious, despite Ava assuming all liabilities. Our path takes us through some rural dwellings, and a horse and carriage pass by. When we reach the opening to the tunnels, we find our paths have crossed with the main party who are just emerging from them. We return along the ridge, pausing to listen to a toad orchestra that had formed in the river, while Dominic takes an impromptu Romanian lesson from one of the guides.

Unfortunately this is the only group photo we have here. Despite George having consumed the most pizza yesterday, it is Thomas and Patrick who put it to best use, having gained a foot in height overnight.

Brian gazes thoughtfully into the middle distance while the other IMO team members are faced with an ethical dilemma.

After lunch we play volleyball, and by the end we just about reach the point where having serve becomes an advantage as opposed to merely having the first opportunity to screw up. We soon return to the hotel to prepare for the closing ceremony. The guides can't seem to decide what time we need to be back at the school at, seemingly citing every possible time between 5 and 6pm. We are pleased to arrive at the correct hour, leaving time for photos and another interview with the local TV station.

The closing ceremony is another well-run affair, keeping well within the 90 minutes that had been allotted to it on the timetable. The five UNK medal-winners claim their medals, and more of them have the flag the correct way up than the wrong way up which we consider a success. A local voice group leads the Moldovan national anthem, after an agonising last-minute prelude in the form of the first movement of John Cage's *4'33"*. They appear several more times to perform their renditions of other well-chosen songs such as *We Are The Champions.* The 2020 BMO's location is confirmed as Romania, and the Romanian leader announces that Preda Mihăilescu will be present to give a talk, thus solving 2019 BMO Problem 6 in the process.

Alevtina and Brian collecting their medals but solving 2019 BMO P0 incorrectly.

Thomas collecting his medal and the day's first independent correct solution to 2019 BMO P0.

Liam and George collecting their medals, also solving 2019 BMO P0 successfully.

We take more photos with our metallic prizes, then head to the farewell dinner where the national dishes of Moldova are served. We discreetly open and split a bottle of champagne. Alevtina goes back for an extra helping of rice wrapped in cabbage leaves, and the server exclaims 'Wow, somebody likes these!'[2].

The team heads outside to the disco, where each team is given a Chinese lantern to release into the night. Ours very nearly crashes into the side of the school but the wind relents and it manages to clear the roof. The dance floor is uneventful until the UK team collectively request the hotel elevator music be played in all its glory. Slowly but surely the crowd energises and soon everyone forms a large circle and is dancing to music as diverse as *'Baby Shark'* and local folk music. Liam suggests getting the entire disco to invert itself with respect to our circle, but we decide this would lead to an implausibly large density of people inside the circle. Despite our initial apprehension, Dominic informs us that our dancing skills have comfortably bested the UK's 2017 effort and we might even be in contention for the country's best dancing effort ever at a Balkan MO.

We return to the hotel and cards until past midnight.

Not a single reference to Star Wars is made during the day.

[2] Or whatever that is in Russian

UNK Air's maiden flight

Day 6 – Sunday 5th 2019

We are treated to an 8:30 start, allowing some of us to begin repaying the large sleep debt we have built up over the previous week. Having packed, Ava and the team head to a local cafe for breakfast while Dominic attends church. The food is excellent if a little slow to arrive. Dominic joins us just before we have to leave, meaning he is forced to enjoy his second breakfast in the form of a chocolate croissant on the minibus to the airport.

Upon leaving the hotel, it dawns on us that we will possibly never hear 'Everybody Dance Now' ever again, or at least in the near future. If we do ever hear that song again, it will certainly remind us of our stay here. We take a short pause to mourn our loss.

Once checked in, we learn that the digits in our departure time have been permuted, delaying our 12:30 flight to 13:20, so we find a table and proceed to do what is quite possibly more maths than has been done during the rest of the trip combined.

Problem 7 (Dominic)**.** *N candidates are to be interviewed sequentially for a single job position. After an interview, a decision on whether to accept (and by extension reject all future candidates) or reject the candidate must immediately be made. All candidates have a real-numbered score associated with them, and the interview can determine this value exactly. The aim is for the interviewer to maximise the probability of hiring the candidate with the highest score. What strategy should they use?*

Problem 8 (Dominic)**.** *You have an infinite chessboard and an infinite supply of queens. For which integers n from 0 to 8 inclusive is it possible to arrange a (possibly infinite) number of queens such that every queen is attacking exactly n others?*

Problem 9 (George)**.** *As Problem 8 but with knights instead.*

We tackle Problem 7 with gusto, despite the fact that calculus soon turns up which is known to induce vomiting in a non-negligible proportion of coordinators. Patrick finds a solution but it is found to only work for the case of infinitely many candidates[3]. Problem 8 is quickly subdivided between the team and before long only the $n = 7$ case remains, which is only solved by George and Alevtina around an hour into the flight.

[3] which must be spherical and in a vacuum

At around 12:30 the team contemplates getting some food before the flight, but we notice a large queue forming outside gates 1&2. We realise that 13:20 refers not to the boarding time but to the take-off time. Ava uses the reasoning that we all have checked luggage to conclude that we still have time to buy lunch, which ends up mainly being eaten on the shuttle bus to the plane. It is around this time that Problem 9 is proposed, which ends up being deemed more interesting than Problem 8.

After several more hours of maths (pausing occasionally to turn the page over whenever a ~~Muggle~~ non-mathematician passes by), the plane lands at Stansted. We note that the applause is considerably less enthusiastic than it was upon arrival in Chişinău. We soon have speculation as to the reason as the pilot announces that it is a rather disappointing 8 degrees in London. Our passage through Stansted is smooth, marked only by waiting for Alevtina to pass through the non-EU passport queue. The team say their farewells and disperse, ready to resume the mindless drill of A-level physics revision. Patrick drew the short straw of living near Winchester resulting in him travelling most of the way home with Dominic. . . .[4]

Problem 10 (Dominic)**.** *How do you spell Mihăilescu?*

For Patrick (who opted to defect to the physics olympiad) and Thomas, this marks the end of their maths olympiad journey. They are now free to enjoy life without ever encountering areal coordinates again should they wish. The other four will attend the final selection camp in Leeds where the team of 6 will be selected from 10. Brian will also be available for team selection in 2020. Four of the five year 13s will be attending Trinity College next year (STEP permitting), with only Alevtina having the sense to escape to Princeton.

Scores and Analysis

The UK team's scores were as follows. Boundaries were 31 for gold, 27 for silver and 15 for bronze.

Name	Q1	Q2	Q3	Q4	Σ	Medal
Brian "Silent Queen" Davies	10	0	10	0	20	Bronze Medal
Thomas "3 Same Suit" Finn	10	10	2	0	22	Bronze Medal
Liam "Arithmetic Progression" Hill	10	4	10	0	24	Bronze Medal
George "8 Reverses" Mears	10	4	10	0	24	Bronze Medal
Alevtina "Consecutive" Studenikina	4	5	10	0	19	Bronze Medal
Patrick "King Counts In Twos" Winter	10	1	3	0	14	Honourable Mention

This marks the first time in 3 years that the UK team receive at least five medals at a Balkan MO. Interestingly, the last time a student was one mark short of the bronze boundary was in 2010, when the competition was also hosted in Moldova.

Problems 1 and 3 are well-answered by the team, with no full solutions given anything other than 10/10. Problem 2, being an inequality, is less well-suited to our skills, although there was plenty of partial credit to be had. The same cannot be said for Problem 4, where only ten students in the entire competition scored marks (and the only two students to score 10 were the pair of perfect scorers).

Six students scored 24 marks, including George and Liam, and all of them achieved this via a breakdown of 10 4 10 0. Similarly, fifteen contestants got a total score of 30, and all of them scored 10 10 10 0.

The infamous UNK0, having debuted at the 2016 BalkanMO and thus being the only UK contestant to attend this competition more than once, has a score defined by the value at x=0 of the fifth-degree polynomial $p(x)$ such that $p(n)$ equals UNKn's score for $1 \leq n \leq 6$. This definition also extends to individual question scores. After losing 26 marks on Q1, a further 101 on Q2 and not even attempting Q4, UNK0's sheer geometry skills managed to claw back 137 marks on Q3 to give a total score of 10, a huge improvement over 2016's dismal performance of -89.

[4] The first draft of this report had 'most of the way' omitted but Dominic understandably asked for this clarification to be added!

Our Thanks

The 2019 Balkan MO was a brilliant and well-run experience, and that would certainly not have been the case without a large number of people. In particular, we would like to thank:

- The Lyceum Da Vinci for hosting much of the event, including the exam itself, the closing ceremony and most of the meals
- The jury for putting together a good paper from a rather deficient shortlist
- The organisers for keeping an event involving a large group of mathematicians from many countries running smoothly and without a hitch
- The Dominics for coordinating our scripts and fighting over every partial mark we could get
- Ava for keeping us safe and ~~as~~ sane ~~as a mathematician could possibly be~~ during our stay, and for making sure our complex dietary requirements were met (Alevtina also deserves a shout for acting as an interpreter when explaining these requirements to staff)
- Valeriu for being an extremely helpful guide who made sure we never got lost, whether at 7:30 in the morning or 9:30 at night, and for following around a group of fickle mathematicians who can never decide what they want to do during free time
- Everyone at UKMT for funding the flights, doing all the work behind the scenes and of course running the superb training camps
- The hotel, which was clean and luxurious even if it came with a couple of quirks

And finally, we thank you for reading this report. Everybody can dance now, and you may clap your hands.