Balkan Mathematical Olympiad, Serbia 2018
UK Report

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The Balkan Mathematical Olympiad is a competition for secondary school students organised annually by eleven countries in Eastern Europe on a rotating basis. The 2018 edition was held near Belgrade, Serbia from 7$^{\text{th}}$ until 12$^{\text{th}}$ May. The UK was grateful to be invited as a guest nation.

Our participation is arranged by the UK Maths Trust[2], as part of a broader programme to introduce the country's most enthusiastic young mathematicians to regular problem-solving, challenging mathematics, and several annual opportunities to participate in competitions. For the Balkan MO, we have a self-imposed rule that students may attend at most once, so that as many as possible might enjoy the experience of an international competition.

This year's UK team was

Agnijo Banerjee	Grove Academy, Dundee	(17)
Nathan Creighton	Mossbourne Community Academy	(17)
Alex Darby	Sutton Grammar School for Boys	(17)
Tom Hillman	St Albans School	(16)
Giles Shaw	Bishop Stopford School	(18)
Aron Thomas	Dame Alice Owen's School	(16)

Dr Vesna Kadelburg from The Perse School was co-leader, and Jill Parker accompanied our students. The results of the UK team were:

	P1	P2	P3	P4	Σ	
Agnijo Banerjee	1	10	10	1	22	Bronze Medal
Nathan Creighton	3	10	8	0	21	Bronze Medal
Alex Darby	0	5	3	0	8	
Tom Hillman	0	6	10	0	16	Bronze Medal
Giles Shaw	0	8	0	0	8	
Aron Thomas	10	10	9	0	29	Silver Medal

The cutoffs for bronze, silver and gold medals were 15, 29 and 40 respectively. These were calculated with reference to the 62 contestants from official member countries, with roughly 2/3 of such contestants receiving a medal.

The leading team totals were (with guest nations in brackets): Bulgaria 230, Romania 223, Serbia 193, Turkey 182, (Kazakhstan 179), (Serbia B 155), Greece 135, (Saudi Arabia 123), with the UK on 104 close to several other countries. Particular congratulations to the eleven students who obtained a perfect score of 40/40, and a gold medal. It's also note-worthy that Aron is the youngest British contestant to receive a silver medal at this competition.

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[2]https://www.ukmt.org.uk/

For many of these top-scoring countries, the students will have been doing competitions of this kind since elementary school, while for the majority of our UK team, their first exposure to this kind of mathematics came eight months ago, at our introductory camp in Oxford. Indeed, Giles' first experience came only six weeks ago at our selection camp in Cambridge! So the UK performance is not only creditable, but the preparation and the experience stands the team in good stead for the future, in olympiads, and in mathematics more generally. Agnijo and Giles have offers to study maths in Cambridge next year, at Trinity College and St. Catherine's College, respectively. Before then, Agnijo will be part of the UK team at the International Mathematical Olympiad, as will Tom and Aron. Nathan and Alex have one more year at school, and have an excellent chance to be part of the team when the UK hosts the IMO in Bath in 2019.

The problems

The Balkan MO comprises a single 4.5 hour paper, which contains four problems, one from each of the main olympiad areas. The difficulty range and gradient is slightly more variable than the IMO.

1. A quadrilateral $ABCD$ is inscribed in a circle Γ, where $AB > CD$, and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and the perpendicular from M to AB intersects the segment AB at the point E. If EM bisects the angle CED, prove that AB is a diameter of Γ.

   (Bulgaria) Emil Stoyanov

2. Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the nth minute ($n = 1, 2, \ldots$) each of them chooses whether to walk due north, east, south, or west, and then walks q^n metres in this direction. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q.

   (United Kingdom) Jeremy King

3. Alice and Bob play the following game. They start with two non-empty piles of coins. Taking turns, with Alice playing first, each player chooses a pile with an even number of coins and moves half of the coins of this pile to the other pile. The game ends if a player cannot move, in which case the other player wins.

   Determine all pairs (a, b) of positive integers such that if initially the two piles have a and b coins, respectively, then Bob has a winning strategy.

   (Cyprus) Demetres Christofides

4. Find all primes p and q such that $3p^{q-1} + 1$ divides $11^p + 17^p$.

   (Bulgaria) Stanislav Dimitrov

Commentaries on the problems

The following commentaries on each problem are not supposed to be official solutions, though they do include solutions, or substantial steps of solutions. I've tried to emphasise what I feel are the key ideas, and how one might have arrived at them naturally, though both stages of this are highly subjective. Any potential olympiad students will find it more valuable to try the problems themselves *before* reading any of this section, and I've also included some exercises and partial problems for you to think about.

It might well be the case that these questions are in the correct order for top students completing a high school education in the *mathematical gymnasia* of Eastern Europe. But perhaps for interested typical British students, the following order might be more appropriate.

Problem Three

Alice and Bob play the following game. They start with two non-empty piles of coins. Taking turns, with Alice playing first, each player chooses a pile with an even number of coins and moves half of the coins of this pile to the other pile. The game ends if a player cannot move, in which case the other player wins.

Determine all pairs (a, b) of positive integers such that if initially the two piles have a and b coins, respectively, then Bob has a winning strategy.

Clearly, the game ends when both piles are odd. If one pile a is odd, and the other b is even, then only one move is possible, namely ending up $a + b/2$ and $b/2$. It's not possible that both of these are odd, so further analysis would be required. However, we might notice from this that if a is even, and b is 2 modulo 4, then there are two possible moves, but at least one of them ends up with both piles now being odd.

So when the official solution starts with the sentence 'let $v_2(a)$ be the exponent of the largest power of two dividing[3] a', this is not magic, but a natural response to a preliminary investigation along the lines of the previous paragraph.

One should then consider some cases. It is clear that Bob wins if (a, b) are both odd, that is $v_2(a) = v_2(b) = 0$, and in our preliminary exploration we established that Alice wins if $a \equiv b \equiv 2$ modulo 4, that is $v_2(a) = v_2(b) = 1$. It's not too hard to establish from here that if $v_2(a) = v_2(b)$, then Bob wins iff this common valuation is even, and Alice wins when it's odd. It's also worth noting that this holds irrespective of the players' choices of moves.

To finish the problem, we now have to classify the remaining cases, and prove what happens in these cases. From the final preliminary observation, we know that Alice wins if $v_2(a) = 1$ and $v_2(b) \geq 1$, but it seems like the game might go on for ever if both players aim to avoid losing when starting from $v_2(a) = 0$ and $v_2(b) \geq 1$. One can try some more small examples, or move straight to a conjecture, but the parity[4] of $\min(v_2(a), v_2(b))$ determines the outcome. In neither case does Bob win, but Alice wins when this minimal valuation is odd, and the game continues forever if it's

[3] This is often called a *valuation*, and is a useful property to consider in many contexts.

[4] *Parity* means 'whether a number is odd or even'.

even, and if you haven't already, you should try proving this by considering how the valuations could change on any move.

As a slight alternative, especially once you know the answer and have observed that the outcome is invariant under multiplying both a, b by four (and so $v_2(a) \mapsto v_2(a) + 2$), one could attempt the following argument. Introduce the notation (a_t, b_t) for the pile sizes at time $t \geq 0$, so $(a, b) = (a_0, b_0)$. We know the outcomes in all cases where $\min(v_2(a), v_2(b)) \leq 2$. So if we start the original game G from a pair (a, b) satisfying $\min(v_2(a), v_2(b)) \geq 2$, we could consider an alternative game G' whose rule for winning instead says that we wait for the first time t such that Alice is to move and $\min(v_2(a_t), v_2(b_t)) \leq 2$. Then we declare the winner (or non-winner) to be the outcome of the original game G started from (a_t, b_t). While the outcome profile is obviously the same as the original game G, we can claim that playing G from (a, b) is the same as playing G' from $(4a, 4b)$, and thus derive the entire outcome profile by induction.

The details required to establish this claim are easy but numerous, and certainly need to be present in a full solution, which explains Alex's unfortunate mark for this problem despite having this sophisticated and workable idea. Finishing the details would be an excellent exercise for anyone aiming to tighten up their combinatorial clarity at this level.

Problem Two

Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the nth minute $(n = 1, 2, \dots)$ each of them chooses whether to walk due north, east, south, or west, and then walks q^n metres in this direction. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q.

The answer is that only $q = 1$ is possible, and the majority of approaches will eliminate all but a finite number of potential values first, then study the cases $q = 2$ and $q = 1/2$ separately. Even though it might seem obvious, remember that you have to provide an example for $q = 1$ too!

This is really a question about polynomials, where the variable is q. So for example, if ant A follows the path NNESWN, then its coordinates after the sixth minute are

$$(x_A^6, y_A^6) = \left(q^3 - q^5,\, q + q^2 - q^4 + q^6\right).$$

So if we want to prove it's impossible for $(x_A^n, y_A^n) = (x_B^n, y_B^n)$ for some different length-n paths, we could first focus on just one coordinate, say the x coordinate. But note that $x_A^n - x_B^n$ is a polynomial in q with degree at most n, where all the coefficients are $\{-2, -1, 0, 1, 2\}$. So if the ants are in the same place at time n, then q is a root of this polynomial.

Insisting on converting q into $\frac{a}{b}$ at an early stage is a sort of intellectual comfort blanket that's probably going to distract from the main insight. But at this stage, we do need to introduce this, and argue that if $q = \frac{a}{b}$ in lowest terms, then q cannot be a root of such a polynomial if either a or b is at least 3. Proving this yourself is definitely a worthwhile exercise. Remember to use that a and b are coprime! (With an additional idea, you can reduce instead to a polynomial with coefficients in $\{-1, 0, +1\}$, from which you can finish even faster.)

To reduce the number of cases left, we can show that there are examples for q iff there are examples for $1/q$, arguing either via the polynomial description (much easier with q rather than $\frac{a}{b}$ again here), or more combinatorially in terms of reversed ant paths.

To finish the problem we have to eliminate one of the possibilities $q = 1/2$ and $q = 2$ (as one follows from the other by the previous paragraph). For $q = 2$, we should study the first time at which the ants diverge, but life will be easier if we argue that we may assume that this happens on the first step. Now, we study the first couple of moves.

- If one ant moves horizontally and the other moves vertically on the first move, then what can you say about the parity of each ant's coordinates after the first step, and indeed after all future steps? This will show that they cannot ever meet.
- Otherwise, assume that both ants move horizontally, one East, one West. Since we can't use parity, but powers of two are deeply involved, it makes sense to consider using congruence modulo 4. Indeed, after this first step, the ants' x-coordinates are not congruent modulo 4 (since one is 1 and the other is -1).
  - If they both move vertically on the second step, or both move horizontally on the second step, this remains the case. (One should check both options for the horizontal case.) Thereafter, all moves have length divisible by 4, and so this property holds forever, and so the ants do not meet.
  - If one moves horizontally, and one moves vertically on the second step, what can you say about the ants' y-coordinates modulo something relevant?

If you want to study $q = 1/2$ instead, you might observe by trying some examples that if the ants head off in different directions, there is a real sense that they become *too far apart* to get back together using the future allowed moves. This motivates considering some sort of distance argument. The interplay between the coordinates is not really suited to standard Euclidean distance, since the ants can't walk in a diagonal direction (which is what will mostly determine the Euclidean distance). Instead, it's worth studying $d_n(A, B) := |x_A^n - x_B^n| + |y_A^n - y_B^n|$ (which is sometimes[5] called the *taxicab distance*.) What is $d_1(A, B)$, and can you control $d_n(A, B) - d_{n-1}(A, B)$ strongly enough to show that $d_n(A, B)$ is always strictly positive? If you can, perhaps you can draw an analogy with the argument for $q = 2$ as a final insight into the workings of this interesting question?

Problem Four

Find all primes p and q such that $3p^{q-1} + 1$ divides $11^p + 17^p$.

None of the UK students solved this problem during the competition, but several managed it during some free time the following morning. Nathan's solution, lightly paraphrased, will follow shortly.

In a question like this, you don't know how many of the details will be crucial. Is the choice of $\{3, 11, 17\}$ going to be important? How will we use the fact that q is prime? You probably can't answer these meta-questions immediately. It also looks like standard motifs of subtracting

[5] Or *Manhattan distance*, or ℓ_1-distance. The motivation for the first is clear is you've seen a map of the rigidly-gridlike layout of Manhattan, where both horizontal and vertical distance are measured in 'blocks', and total distance is the sum of streets and avenues.

multiples of $3p^{q-1}+1$ from 11^p+17^p is not going to make life easier. Nathan's approach is to study the possible factors of 11^p+17^p, focusing on prime power factors. Once he has a rich enough understanding of potential such factors, he can then study whether they combine to form $3p^{q-1}+1$, which turns out to be very restrictive, leaving only a handful of cases to eliminate by hand.

Nathan writes: We can quickly eliminate the possibility that $p=2$, and so now assume we have a solution where p is odd.

Claim I: None of 8, 49 or 11 divide $3p^{q-1}+1$.

Proof. It's enough to show that they do not divide 11^p+7^p. The non-divisibility of 11 is clear. For 8, note that $11^p \equiv 1,3$ and $17^p \equiv 1$ modulo 8, and so $11^p+17^p \equiv 2,4 \not\equiv 0$.

To handle 49, we rewrite 11^p+17^p as $11^p-(-17)^p$ and we have the valuation $v_7(11-(-17)) = v_7(28) = 1$. So when we *lift the exponent* (see later), we find

$$v_7\left(11^p-(-17)^p\right) = 1 + v_7(p).$$

So if $49 \mid 11^p+17^p$, then the LHS is at least two, and so $v_7(p) \geq 1$. But then $p=7$ is the only option, for which certainly $49 \nmid 3p^{q-1}+1$. The claim is now proved. □

So we may now write

$$3p^{q-1}+1 = 2^a 7^b \prod r_i^{e_i}, \tag{1}$$

where r_i are primes not equal to $\{2,7,11\}$, and $a \in \{1,2\}$, $b \in \{0,1\}$.

Claim II: each $r_i \equiv 1$ modulo p.

Proof. As before $r_i \mid 11^p-(-17)^p$, and since $r_i \neq 11$, 11 has a multiplicative inverse modulo r_i, and so indeed there exists t such that $11t \equiv -17$ modulo r_i. Using this in the divisibility relation:

$$r_i \mid 11^p-(-17)^p \equiv 11^p-(11t)t^p \equiv 11^p(1-t^p) \quad\iff\quad r_i \mid 1-t^p.$$

The order of t modulo r_i then divides p, so is either 1 or p. If this order is 1, then $t \equiv 1$, but then, modulo r_i, $11 \equiv -17$, so $r_i \mid 28$, which we have excluded already. So the order is p, and thus $p \mid r_i - 1$, as we claimed. □

Going back to (1), we have

$$1 \equiv 3p^{q-1}+1 = 2^a 7^b \prod r_i^{e_i} \equiv 2^a 7^b \qquad \bmod p,$$

and so $p \mid 2^a 7^b - 1$. But remember that $a \in \{1,2\}$ and $b \in \{0,1\}$, so there are only a handful of cases to check. Each of the other cases requires a line or two to eliminate, so do try this yourself! In the end, though, we see that $(a,b)=(2,0)$ or $(2,1)$, both leading to $p=3$ are the only possibilities. Returning to the original question, we just have to check possible solutions to $3^q+1 \mid 11^3+17^3 = 2^2\cdot 7\cdot 223$, which we can do manually (for example by checking all prime $q \leq 7$), to find that the only solution is $(p,q)=(3,3)$.

Dominic: As part of this solution, Nathan uses the *lifting the exponent lemma* to control $v_7(11^p - (-17)^p)$. This example is simple enough that it's probably easiest to go directly. Since p is odd, we can factorise

$$11^p + 17^p = 28 \cdot \left(11^{p-1} - 11^{p-2} \cdot 17 + 11^{p-3} \cdot 17^2 - \ldots + 17^{p-1}\right).$$

Can you come up with an argument for why 7 cannot divide the second factor? Some of the notation Nathan used elsewhere in his solution may be useful! If you can, then you've shown that $v_7(11^p + 17^p) = 1$.

The general statement of the lemma relates $v_p(x^n - y^n)$ to $v_p(n)$ and $v_p(x - y)$, which explains why Nathan converts $+17^p$ to $-(-17)^p$, though it makes little difference to the proof. You can find statements of this lemma, which has become relatively well-known recently in this community (and which is sometimes attributed to Mihai Manea), in many places on the internet and in modern books. The proof is very similar in the general case to the special case discussed previously. It's worth remembering that the case $p = 2$ always requires extra care (and indeed a different statement). This distinction comes from the fact that the simultaneous congruence equations $x + y \equiv 0$ and $x - y \equiv 0$ modulo n have two pairs of solutions when $2 \mid n$.

It's worth noting also that in a solution like Nathan's where different ranges of options are excluded one after the other, this clear organisation into claims is of huge benefit to the reader, irrespective of how much text is or isn't included as a prelude.

Problem One

A quadrilateral $ABCD$ is inscribed in a circle Γ, where $AB > CD$, and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and the perpendicular from M to AB intersects the segment AB at the point E. If EM bisects the angle CED, prove that AB is a diameter of Γ.

I do not think that this was the hardest question on the paper, but I have the most to say about it, so it comes last here. The section entitled 'Step One' contains (including the exercise at the end) a complete solution which only uses familiar material. The remaining sections have to quote some more obscure material, and may be of less interest to inexperienced readers, for whom many other Balkan and IMO geometry problems might be more appropriate.

Although I've been working hard to improve my geometry over the past couple of years, my attitude to the subject remains recreational. I prefer problems with a puzzle-like quality rather than this sort of question, whose statement is, after a little thought, not so surprising, even if most proof methods are either complicated (but elementary) or exotic. I feel most approaches to this problem require three steps: it's easy to read a solution and forget that the first step really is a step!

I'm fairly vigorously opposed to software diagrams, as at least for me they discourage exactly the sort of insights one is generally hoping for. If you are reading this section carefully, you can find hand-drawn diagrams on my blog[6], but almost certainly the most useful method is to draw your own. There are only five points, though you might like to peek at Step Zero to inform drawing an accurate enough diagram without needing to apply the condition by eye.

[6]available shortly: `https://eventuallyalmosteverywhere.wordpress.com/olympiad`

Step Zero: Introduce X, the intersection of AD and BC

To follow through any synthetic approach, it's essential to have a good perspective on what the diagram 'means', and you will almost certainly need to introduce X to get such a perspective. Here are a couple of reasons why you might think to introduce X:

- If the conclusion is true, then $\angle ADB = \angle ACB = \pi/2$, and so M lies on two altitudes, and thus is the orthocentre of some triangle. Which triangle? It's $\triangle AXB$.
- Alternatively, the corresponding altitude is an angle bisector of the pedal triangle, and so the given diagram might remind you very strongly of this. Which triangle has pedal triangle $\triangle CED$? It's $\triangle AXB$ again.
- If your diagram was accurate enough (and since part of the statement is a 'given...' this is not so easy) you might have noticed that AD, ME and BC were concurrent. Where? At $X := AD \cap BC$, obviously.
- In a similar vein, if the conclusion is true, then $ADME$ and $BEMC$ are both cyclic, and we are given $ABCD$ cyclic. The radical axes of these three circles are AD, ME, and BC, so it is reasonable to guess that X, the (hypothesised) point of concurrence is relevant. See later.
- You are given part of a complete quadrilateral (since M is one of the intersection points of quadrilateral $ABCD$) - it might well be useful to complete it!
- Random luck. It's not unreasonable to consider arbitrary intersections, though this can be a low-reward strategy in general. If you did introduce X for no reason, you then had to guess, observe or realise that X, M and E should be collinear.

Step One: Proving X, M, and E are collinear?

This is harder than Step Two I think, so is postponed.

Step Two: showing the result, given X, M, E collinear

The official solution proposes introducing the reflection of A in E, which is certainly a good way to get lots of equal angles into useful places rather than not-quite-useful places. However, probably one didn't spot this. Whether or not this was your motivation in the first place, once X is present, it's natural to look for an argument based on the radical axis configuration. Our conclusion is equivalent to showing that $ADME$ or $BEMC$ are cyclic, and obviously $ABCD$ is given as cyclic.

However, motivated by the radical axis configuration[7] let E' be second intersection of circles $\odot ADM$ and $\odot BMC$. We know that E' lies on line XM, and so it suffices to show that $E' = E$. But by chasing angles in the cyclic quadrilaterals involving E', we find that if $E \neq E'$, then $\angle EE'A = \angle BE'E$, and so $\triangle AEE' \equiv \triangle BEE'$, which after a bit of thought implies $\triangle AXB$ is isosceles, which contradicts the given assumptions.

[7] Which you can look up - but I recommend not getting distracted by what *radical axis* means at this stage. It's a theorem concerning when three pairs of points form three cyclic quadrilaterals, and it has a valid converse! I also recommend not drawing any circles when thinking about the diagram.

Step One: Proving X, M, and E are collinear

By introducing enough extra notation and additional structure, one can prove this part by similar triangles. I think a natural approach in a question with significant symmetry is to use the sine rule repeatedly. This has pros and cons:

- Disadvantage: it's easy to get into an endless sequence of mindless calculations, which don't go anywhere and leads more towards frustration than towards insight.
- Advantage: one can plan out the calculation without actually doing it. Imagine, to give a completely hypothetical example, trying to plan such an approach in a lurching Serbian minibus with only one diagram. You establish which ratios can be calculated in terms of other ratios, and wait until you're back in a quiet room actually to do it.

You might try to show that $\angle ADB = \angle ACD = \pi/2$ directly by such a method, but I couldn't make it work. I could plan out the following though:

- Start with some labelling. I write α, β for $\angle XMD$ and $\angle CMX$, and a, b for $\angle DME$ and $\angle EMC$. The goal is to prove that (a, α) and (b, β) are complementary by showing that $\frac{\sin\alpha}{\sin\beta} = \frac{\sin a}{\sin b}$. Will also refer to $\hat{A}$ for $\angle BAD$ when necessary.
- The first ratio of sines is the easier one. Using the equal length MX in $\triangle DXM, \triangle CMX$, and then the sine rule in $\triangle DXC$, obtain $\frac{\sin\alpha}{\sin\beta} = \frac{DX}{CX} = \frac{\sin\hat{A}}{\sin\hat{B}}$.
- We can obtain $\frac{\sin a}{\sin b} = \frac{DE/DM}{CE/CM}$, but this could get complicated. However, by exploiting the equal angles $\angle DEA = \angle BEC$, we can derive $\frac{DE}{CE} = \frac{AD}{BC}\frac{\sin\hat{A}}{\sin\hat{B}}$. But of course, $ABCD$ is cyclic, and so there are relevant similar triangles, from which $\frac{AD}{BC} = \frac{DM}{CM}$. So in fact we have shown $\frac{\sin a}{\sin b} = \frac{\sin\hat{A}}{\sin\hat{B}}$, as we wanted since now we know.

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin a}{\sin b}. \tag{2}$$

- We need to be careful as this doesn't immediately imply $\alpha = \pi - a$ and $\beta = \pi - b$. (For example, we need to exclude $\alpha = a$! It's useful to exploit the fact that both a and b are obtuse here. For this type of thing, it's more useful to focus on showing uniqueness (we definitely know one solution!) rather than *finding* all solutions. We are essentially asked to show uniqueness of a solution to an equation like

$$\frac{\sin(\theta - x)}{\sin x} = z, \tag{3}$$

where $\theta < \pi$. After suitable rearranging, (3) determines $\tan x$, and so certainly has at most one solution in any interval of width less than π. This is a standard issue when using this type of argument and it's important to know how roughly how to resolve such issues, as you wouldn't want to waste significant competition time on such technicalities.

As an exercise, you can try to prove Step Two using this method. A hint: suppose M is *not* the orthocentre of $\triangle AXB$. Introduce points C', D' such that $\angle AD'B = \angle AC'B = \pi/2$. Now AE bisects $\angle DEC$ but also $\angle D'EC'$. Can you use this to find two congruent triangles which can't possibly actually be congruent?

An alternative synthetic approach

UK student Alex started with the following observation. Simple angle-chasing in cyclic quadrilateral $ABCD$ reveals that

$$\pi/2 - \angle AME = \angle EAM = \angle MDC, \quad \pi/2 - \angle EMB = \angle MBE = \angle DCM. \tag{4}$$

But we are given that M lies on the angle bisector of $\angle CED$. So we make the following claim.

Claim: the only point M which lies on the angle bisector and satisfies (4) is the *incentre* of $\triangle CED$.

Remark: This claim is false. However, it is true that such a point can only be the *incentre* or E*-excentre* of $\triangle CED$. One could salvage the original by restricting M to lie inside the triangle.

Remark: As was heavily discussed, this claim is certainly not well-known. It is very believable, but it is also not obvious either. An approach by ratios of sines, for example, as in the solution given above, seems rather tricky. Aron's argument below is lovely, but again 'brief $\neq$ easy'!

Proof of claim (Aron): Write $\theta := \angle MDC$ and $\varphi := \angle DCM$. Consider the altitude MX in $\triangle MDC$. This is isogonal in this triangle to line ME, because the angles $\pi/2 - \theta$ and $\pi/2 - \varphi$ are interchanged at M. This means that the circumcentre of $\triangle MDC$ lies[8] on ME. But the circumcircle of $\triangle MDC$ also lies on the perpendicular bisector of CD, and this meets the angle bisector on the circumcircle of $\triangle CED$. Indeed, this intersection point is the arc midpoint of CD, and this really is well-known to the circumcentre of $\odot ICI_E D$, the circle which includes the incentre and the E-excentre, and so this characterises the two possibilities for M, as required.

Harmonic ranges

In the end, the most straightforward approach to this question was to use harmonic ranges. Personally, I would use this to complete what I referred to as Step One, namely showing X, M, E collinear. I feel the radical axis argument given above is a more natural way to handle the second step, though one can also deploy projective theory for this too in relatively few steps.

This is not the place for an in-depth introduction to harmonic ranges. However, I think less experienced students are often confused about when they should consider looking for them, so I'll try to focus on this.

What is it? Study four points A, B, C, D on a line ℓ, grouped into two pairs $(A, B), (C, D)$. Then define the *cross-ratio* to be

$$(A, B; C, D) := \frac{\overrightarrow{CA}}{\overrightarrow{CB}} \div \frac{\overrightarrow{DA}}{\overrightarrow{DB}}. \tag{5}$$

We say that $(A, B; C, D)$ form a *harmonic range*[9] if their cross-ratio is -1. This certainly implies that one of (C, D) lies between A and B, and the other lies outside. Note that this is a property of *two pairs of points*, not of four points! $(A, B; C, D)$ harmonic does not imply $(A, C; B, D)$ harmonic and so on. Crucially, there is an analogous definition for two pairs of points lying on a given circle.

[8] Perhaps you are more familiar with the stronger statement that the orthocentre and circumcentre - eg of $\triangle MDC$ - are isogonal conjugates?

[9] Or *harmonic bundle*, *harmonic system*, etc etc.

What can you do with harmonic ranges? There are two reasons why they are useful in solving geometry problems:

1. They often appear in standard configurations and given configurations!
2. Given one harmonic range, there are natural ways to generate other harmonic ranges.

We'll discuss both of these in a second, but a rough outline of a typical proof using harmonic ranges is as follows. First, identify a harmonic range in the configuration, perhaps using a standard sub-configuration; then, project this first harmonic range around to find some new, perhaps less obvious, harmonic ranges; finally, use some converse result to recover a property about the diagram from your final harmonic range.

We need to discuss the two useful reasons given above in more detail:

1. Take a triangle $\triangle ABC$, and consider the intersection points D, E of the internal and external A-angle bisectors with the opposite side BC. Can you prove (for example using a theorem about lengths in the angle bisector configuration...) that $(B, C; D, E)$ is harmonic?

   A related example occurs when you have both Ceva's configuration and Menelaus's transversal present in a given triangle, as you then have a harmonic range too. (See the suggested notes.)

   One of the points may be the *point at infinity* on ℓ. Without getting into philosophy, can you see how to choose C so that $(A, B; C, \infty)$ is harmonic? This is a very very useful example.

   There are plenty of good examples for cyclic ranges too, which you can explore yourself.

2. Harmonic ranges live in the world known as *projective geometry*. What this means in general is not relevant here, but it's a good mnemonic for remembering that one can *project* one harmonic range to acquire another. The most simple example is this.

   Given A, B, C, D on a line ℓ, let P be some point not on ℓ. The set of lines (PA, PB, PC, PD) is often referred to as a *pencil*. Now, consider intersecting this pencil with a different line ℓ' (again not through P) to obtain a new set of points (A', B', C', D'). The key fact is that if $(A, B; C, D)$ is harmonic, then $(A', B'; C', D')$ is also harmonic!

   Not only does this give a new harmonic range, it establishes that the harmonic property really depends on the pencil of lines, rather than the choice of ℓ. Letting ℓ vary, we get an infinite collection of harmonic ranges. So if your diagram has a suggestive pencil of four lines, this is a promising sign that harmonic ranges may have value.

   One can also project between lines and circles and from circles to circles, and typically you will need to do this.

How do you prove the results? If you proved the first example above using the angle bisector theorems, you might ask 'how do you prove the angle bisector theorem'? Well, there are elegant synthetic methods, but the sine rule is a fail-safe mode of attack too. Essentially, almost all results about harmonic ranges can be proved using the sine rule, perhaps with a bit of help from other standard length-comparison results, in particular Menelaus, Ceva, and trigonometric Ceva.

As we've seen in the first attempt at Step One, sine rule calculations can be arduous. Projecting harmonic ranges can be a *shortcut* through such calculations, provided you know enough examples.

How do I know when to use them? This is really just a reiteration:

- If you are given a configuration and you recognise part of the diagram as a harmonic range, it might well be worth pursuing this. If you can't project it into any useful other harmonic range (even after, for example, introducing one extra intersection point), this might lead nowhere, but you'll probably find something.
- If you see that part of the diagram is well-suited for projecting harmonic ranges into other harmonic ranges, this is relevant. For example, if there are several lines through one point, particularly if that point also lies on a relevant circle.
- Similarly, if you require some sort of symmetric result like 'points $\mathcal{A}$ have some tangency condition iff points $\mathcal{B}$ have the same tangency condition', then consider whether the condition has a harmonic range interpretation, and whether $\mathcal{A}$ can be projected onto $\mathcal{B}$.
- If it feels like the problem could be solved by a giant sine rule calculation comparing various ratios, it might be amenable to harmonic range analysis, so long as you find a first example!

Where can I find actual details? Because this is a report on a contest, rather than a set of lecture notes, the level of detail given here is intentionally very low. Though I hope it gives a useful overview of why such approaches might be useful, perhaps especially for those students who have a passing familiarity with harmonic ranges, but are not yet fluent at successfully applying the methods in actual problems.

The detail is important though, and I recommend these resources, among many articles on the internet:

- Alexander Remorov's sheet on *Projective Geometry*, which also includes a discussion of polars. My own knowledge of the subject is particularly indebted to this source. I like Question 4. The link is in this[10] footnote.
- Sections 9.2–9.4 of Evan Chen's recent book *Euclidean Geometry in Mathematical Olympiads* includes an ideally compact repository of useful statements. Problems, some of which veer into more challenging territory, are at the end of the section.

The stages of the contest

Problem selection

The programme of this competition is a scaled down version of the IMO. The leaders gather in suburban Belgrade on Monday night to select four problems from a shortlist compiled by the organisers. To recreate the students' experience, it makes sense to start by trying these without reference to solutions. Some of the questions are UK submissions, so I can briefly astonish my colleague Vesna with almost instant fluency, before admitting that I wrote or edited the corresponding solutions.

Making the choice occupies Tuesday morning. As always, it feels slightly like a shot in the dark, as one night is not really sufficient to get a feeling for twenty problems, especially the hardest

[10] `http://alexanderrem.weebly.com/uploads/7/2/5/6/72566533/projectivegeometry.pdf`

ones. In the end, there was clearly a unique good hard problem, but unfortunately it had to be rejected because it was too similar to a recent problem from a well-known source. Some of us have been investing considerable energy in finding natural Euclidean arguments to the geometry problem chosen as Q3, but once Greek leader Silouanos outlines the role of harmonic ranges, it is hurriedly moved to Q1. I think the resulting set of four questions are attractive, but with a rather compressed difficulty range, and certainly not in the right order for the UK students, whose geometric toolkits probably don't yet include the ideas needed to access the 'easy' solutions.

In any case, it's interesting to discuss with the leaders from some of the eleven Balkan full member countries. Our opinions differ concerning which styles of problem give an advantage to extensively-trained problems. I personally feel that Q2 and Q3 are accessible even to students (or adults!) without much mathematical background, whereas here is a prevailing view that no problem with combinatorial flavour is ever 'easy'. By contrast, many of the ideas required for a short solution to either Q1 or Q4 might be considered obscure even by serious olympiad enthusiasts, though feature on the school curriculum, at least for the most able children, in many of these countries.

We have to finalise the wording of the problems, and there are many many proposed improvements to Q2 and Q3. The final problem, unsurprisingly, requires considerably less attention. That's our job done for the British delegation, while the other leaders get to work producing versions in their own languages, including Bosnian and Serbian, the (non-)differences between which can happily fill one dinner's worth of interesting conversation.

The contest

On Wednesday morning, we are transferred to the contestant site, in the rolling hills just outside the south-east city limits of Belgrade. An extremely brief opening ceremony takes place in a room slightly smaller than the number of people attending the competition. The UK team look happy enough perched on a table. Two local violinists play Mozart with a gypsy flourish, before Teodor von Burg, a former Serbian olympiad star and graduate of Exeter College, Oxford, speaks briefly about the usual clichés of such speeches, and the additive paradox of wishing *everyone* good luck before a competition, then ends rapidly to avoid indulging such clichés himself.

After the contestants fan out to various exam rooms spread through the hotel, the contest begins and they are allowed to ask queries about the problems for 30 minutes. Many many students ask 'what does *exactly the same route* mean?' and 'what if Alice and Bob play forever?', but some variety is provided when Aron shares his detailed dilemma about the exact usage of carbon paper[11].

After Monday's 2am start, I am overdue a nap. There has been some room-swapping, and mine is reserved for 'Professor Mr Jill Parker'. Whomever the bed truly belongs to, I leave it in time to meet the team outside the exam with Jill and Vesna. As we'd predicted, many are enthusiastic about Q2 and Q3, but have been frustrated by the geometry. Tom crowd-sources an investigation to recover a result about the incentre claimed by Alex, who perhaps now regrets, in his rush to move to other questions, not offering more of such details himself. No-one claims anything beyond observations in the number theory, so we suggest they keep thinking about it through the afternoon.

[11] Future UK students: this is not to become a habit, please...

A brief excursion

Agnijo and Nathan had done their research on Belgrade, and had asked about the possibility of visiting the Nikola Tesla museum. The team have a guide, Sandra, a maths undergraduate, and I'm extremely impressed that she and some of her colleagues are able to organise a visit downtown and guided tour of this museum at essentially no notice for them, along with Italy, Bosnia and Azerbaijan. Vesna and I diverge to make a start on marking in a cafe, rejoining in time for the museum, where Giles apparently learns what 'Azerbaijan' is, and we all learn about Tesla's extraordinary life story, and get to see the original Tesla coil (briefly) in action. Agnijo and Tom have been primed with fluorescent tubes, which do indeed glow as lightning surges between the century-old coil and its crowning sphere. Other exhibits, including highlights from Tesla's wardrobe (pre-dating *geek chic*, it would seem), and an imitation ticket from Belgrade to New York, are perhaps less fascinating.

But the roar of 10^6 Volts is still in our ears as we stroll across the city centre, where Alex confidently identifies several churches as the orthodox cathedral they'd visited earlier, and eyes are drawn to the faded but strident protest banners outside the parliament. We choose a restaurant in bohemian Skadarska street, where prices are low, and availability of protein and itinerant accordion players is high. The team are trying to be polite about their hotel's food, but I sense this variation is welcome. Giles pokes gingerly at a deep-fried pork slab, which erupts with multiple cheeses. The 'Serbian sword' could be retitled 'as many meat items on a stick as possible (plus 1/8 of a pepper)'.

We return to Avala feeling sleepily satisfied. Tom and Agnijo discuss the GCSE question 'prove using algebra that the product of two odd numbers is odd', and whether you can or should prove it without algebra. The taxis clearly sense our post-prandial vulnerability, and operate a creative attitude to receipts, and to powers of ten. But this round of ambiguous paperwork and mathematical corrections is just the prelude for Vesna and myself, who have a cosy night in with the scripts.

Coordination

At a competition, the leaders of each team study their own students' work, and agree an appropriate mark with a team of local coordinators. The UK has an easier workload: we do not have to provide translations, since our students write in English, though some of them might like to note that in a question about parity, mixing up the words 'odd' and 'even' as if flipping a coin does make it harder to convince the reader you know what you're talking about.

We start with 9am geometry, where the coordinators are proposing giving Aron 8 or 9 out of 10 as part of a crusade against citing configurational properties as 'well-known'. Aron has, in fact, outlined a proof of his (fairly) well-known fact, and if the proposal is to award 6 or 7 without this, then the marking team's entire day is guaranteed to be a continuous series of wars. I think the penny drops shortly after our meeting, and Aron gets upgraded to 10/10 at 9.30. Unfortunately, what remains of the crusade will deny Alex any credit at all for his unjustified claim about the incentre, despite its role in an appealing synthetic solution.

The middle two questions have a wide range of arguments. The British work on Q3 is actually pretty good, and even in the two scripts with small corners missing have organised their cases very clearly, and the coordinators (who initially want to give all full marks) can see that the students already had the ingredients to fix their minor errors. Q2 is more challenging. Once we have worked out where the good bit begins, Nathan's solution is clearly superb, and once we've worked out which of his mysterious side-comments to ignore, Giles has all but the final details of a really imaginative solution, and Agnijo is flawless. Aron seems keen to make an *even* number of really confusing mistakes on this paper, so on this question has mixed up 'horizontal' and 'vertical' as if flipping a coin, though the coordinators are more sympathetic than I would have been. Tom claims that his solution is 'very poorly written', which is very far from the case, but after rolling back and forth through his logic a few times, we agree that a couple of cases of q are inadvertently missing.

The students return from their short excursion in time to hear their scores before dinner, and though Alex is a bit disappointed about the non-acceptance of his 'lemma', everyone is broadly pleased with themselves, as they should be. I get my first experience of the infamous hotel salad, which the students had previously described as 'vinegar topped with lettuce', which is roughly accurate, though the rest is nice enough. Agnijo is worried the main course includes beef, but is satisfied with the supposedly vegan alternative, namely a grilled fish.

The Balkan countries take the table of scores a bit more seriously than we do, and so this year's celebratory table is sipping Bulgarian cognac washed down with Romanian tears, though this wholesome rivalry shouldn't distract from the hugely impressive seven perfect scores from those countries' contestants (plus four from the others). The competition at the adjacent table seems to be the relative merits of Serbian, Macedonian and Montenegrin wine and *rakija*. Meanwhile, the UK students have made plenty of new friends to induct into their favourite card games, and some Albanians, Bosnians and Turks seem a) very keen to practise their excellent English, and b) appropriately baffled by the rules, and lack of rules.

Round and about

The bulk of Friday is set aside for an excursion. Our destination is Valjevo, a town two hours' drive west of Belgrade, which represents some sort of historical home for the Serbian maths enrichment community. We gather in their gloriously rococo hall to listen to an in-depth presentation concerning many aspects of daily life at Valjevo Grammar School. The nearby research institute in leafy Petnica offers a more science-focused perspective. The students get to tour some labs, though they don't get to practise for their upcoming A-level or Highers physics by trying any experiments. Nathan, however, finalises his solution to Q4 from the contest, which seems a good use of time, and which you can read earlier in this report. Aron asks me to solve what seems a challenging geometry question in my head. I cannot. A stamp-sized freehand diagram on a napkin doesn't help either.

Vesna was a regular visitor to Petnica as a teenage olympiad contestant, and she has briefed me on the charms of a nearby cave, apparently a regular choice for planned and unplanned excursions during her selection camps in the 90s. The UK group plans to sneak away from the third phase of the tour to find this cave, but we are foiled because the third phase of the tour is indeed a visit to the cave. This involves a short walk, during which Agnijo is harassed by the world's least threatening dog. The temperature is pushing 30C, but Aron is worried about sunburn, so is reluctant to remove his polar fleece. He gently roasts, while Alex tells us some horror stories from his experience as a Wimbledon ballboy during the 2016 heatwave. The cave provides cool relief, and is indeed giant, with plenty of sub-caves underneath the looming stalactites.

It turns out we are in the less impressive half of the cave. The students want to climb to the more impressive upper cave. It may be more impressive, but it is also considerably darker, and I admire Giles' and Nathan's tenacity to find out exactly how far a distant rocky staircase extends into the gloom temporarily illuminated by a phone torch. That concludes the adventure, and we return to Belgrade coated in varying quantities of cave detritus. The return journey affords great views of the distant mountains towards the Bosnian and Montenegrin borders, though Tom is keen to use the time to make a start on coordinating the multi-author student report. Unable to avoid eavesdropping on the discussion, sounds like it will be a substantial document when completed[12]...

Finishing up

Back in Avala, the closing ceremony takes place during dinner, and is informal. Jury chair Zoran Kadelburg awards the certificates; chief organiser Miljan presents the medals; and Miljan's wife notices and steps into the essential role of helping the medallists flip their newly-acquired prizes in front of any flags they might be carrying for the waiting photographers. This one-at-a-time low-key arrangement was actually very nice for everyone, and our four medallists enjoyed their moments.

It is a balmy evening, so we drift outside again. Aron is random-walking, hunting for the WiFi sweetspot so he can download the punchline to our colleague Sam's claimed $\mathbb{C}$omplex solution to Q1 before Nathan finishes rounding up new players for the next round of card games; while Giles and Alex disappear off towards the most distant unlit car park with a troupe of guides and Bosnians and a volleyball. At the leaders' table, Vesna and the other Balkan residents give a collective hollow laugh on hearing that I have elected to travel to the Montenegrin Alps by bus. But that ten hour experience starts tomorrow, outside the remit of this report, which will end here.

Conclusions and thanks

Our six British students benefited greatly from the chance to attend BMO 2018, and everyone seemed to have an excellent time. We remain very glad to have been invited! Among the many people who made this competition a success, I'd like to offer particular thanks to:

- The Serbian Mathematical Olympiad, who put together an excellent competition, which ran flawlessly from the first jury meeting, through to the presentation of the final medal;
- The problem authors, and the problem selection committee, led by Dušan Djukíc, who compiled an attractive shortlist of problems, which included the excellent final paper;
- All the guides had a superb attitude, and the UK was particularly grateful to Sandra for all her efforts to make our contestants' time in Serbia as interesting as possible. We wish her and her friends every success in their mathematical studies;

[12]It is now complete, and can be found towards the bottom at `https://www.imo-register.org.uk/reports.html`

- Everyone behind the scenes at UKMT, particularly Bev Detoeuf, and the staff at recent camps who helped prepare the students so they could get the most out of this experience;
- Jill Parker, who looked after the British students with her usual calm and kindness;
- Vesna Kadelburg, who was our leader in 2007 when I was a contestant at this competition, and who is just as excellent as a colleague as a leader. The collaboration on deciphering combinatorics at 4am, and interpreting idiomatic Serbian menus was both fun, and appreciated;
- Finally, our UK team consisting of Agnijo, Nathan, Alex, Tom, Giles, and Aron, who all made good progress through the competition and solved plenty of other problems during our time in Serbia, and were excellent ambassadors for UKMT, and we're sure they'll enjoy plenty of further success in maths competitions and more generally in the future.

Dominic Yeo
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