Balkan Mathematical Olympiad - Students' Report

Written by Maria Holdcroft and Oliver Feng, with lots of corrections and solutions by Freddie Illingworth and a nice inequality solved by Warren Li.

July 10, 2013

Introduction

This report details the events of the 30th Balkan Mathematical Olympiad, which was held from 28th June - 3rd July 2013 in Agros, Cyprus, from the student perspective. It aims to mention more maths problems than Gabriel Gendler's report [4] for the same event held in 2012, whilst still describing the non-mathematical events that occurred.

As far as the organisation of this year's Balkan MO is concerned, the situation was highly complex and fraught with difficulties right from the start. The event was originally scheduled to take place in Tirana, Albania in the second full week of May, but due to a combination of unfortunate circumstances, the organisers' plans never came to fruition. For a few weeks in April, it was unclear whether a (real) substitute competition would be created in order to replace a 'phantom' event that had been confined to the realm of the purely imaginary. Fortunately, the Cyprus Mathematical Society stepped in at the end of April, and the Balkan MO 2013 was organised at very short notice under the auspices of the new Cyprus Minister for Education and Culture, Mr Kyriakos Kenevezos. It was a great success, and we'll always be forever eternally[1] grateful for the sterling work of the Cypriots who went into overdrive for two months in order to achieve this outcome.

[1] Tautology is awesome because it is. - Adam P. Goucher

Dedicated to Adam P. Goucher and his future legacy.

Participants

In keeping with the spirit of previous student reports, we will now give a show of true unashamed British insularity and dissect[2] the performances of the UNKs at the Balkan MO 2013. The following table gives the results of the team:

Code	**Name**	**P1**	**P2**	**P3**	**P4**	**Total**	**Award**
UNK1	Oliver Feng	10	10	0	0	20	Silver Medal
UNK2	William Gao	2	3	1	0	6	
UNK3	Frank Han	10	8	10	1	29	Silver Medal
UNK4	Freddie Illingworth	10	10	10	4	34	Gold Medal
UNK5	Maria Holdcroft	10	10	10	0	30	Silver Medal
UNK6	Warren Li	0	3	10	1	14	Bronze Medal
UNK7	Geoff Smith						
UNK8	Gerry Leversha						
UNK0	Andrew Carlotti						

There are many interesting aspects in this table that we can discuss at great length. The first may as well be how the Cypriot dictionary is ordered A,... Ha, ..., Il, ..., Ho, ... Upon noticing this, we may also note that the surnames of all the students begin with letters between F and L in the alphabet (inclusive). At this point we make a wild conjecture that next year's team will have surnames beginning with letters between A and N, the year after E and L, and then between L and E (i.e. everything but the letters from F to M). Furthermore, everyone who doesn't have a score congruent to 0 modulo 10 can be paired with another team member whose first name begins with the same letter. This includes the leader and deputy leader, who have no score.

There are also a worrying number of connections to the number two, probably catalysed by the BMOS policy that no British student may compete at the Balkans 2 or more times, hence the 2nd best team is selected. For example, the digital sum of Oliver's score is 2, as well as his score having 2 distinct prime factors, one of which is 2, which is raised to the power 2 in the prime factorisation. William has precisely 2 scores which are 2 or more, and was 2 marks away from the bronze medal boundary. Freddie came 2nd in the entire competition. Maria came 2nd out of all the girls in the competition, or 11th overall, and 11 consists of 2 '1's. Warren was precisely 2^{-1} way between the bronze and silver medal boundaries, as well as being one of only 2 bronze medallists to score 10 marks (2 when read in binary) on question 3 (11 in binary, which contains 2 '1's). As if this wasn't enough, the most common medal awarded, both in our team and in the entire competition, was silver, the 2nd prize. The best UK performance came on problem 2, and the total number of points amassed here was 44[3], which has 2 distinct prime factors

[2]Health warning: We've taken British navel-gazing to a new level. Even if you've been conditioned through exposure to previous material, you might want to skip this section. There might be a bit too much here for you to digest.

[3]This was also the cost in euros of a bottle of wine at the Hilton hotel in Nicosia, at which Geoff stayed for 2 nights

and has 2^2 as a factor. Furthermore, if Frank gave 2 of his marks to William and 2 of his marks to Maria (so as to increase William's total on problem 2 by 2 marks), then we would have received 2 gold, 2 silver and 2 bronze medals (the "elusive 'double traffic lights'" distinction [6]). More generally, the lowest score not achieved by any of the 97 contestants in the competition was 22.

Other interesting aspects include that both Oliver and Frank scored the same in BMO2 as they did in the Balkans (with Frank's score also matching the number of silver medals awarded), Maria scored 30 in the 30th Balkan Mathematical Olympiad, the exam of which was sat on the 30th June. Warren is declared an anomalous result, since he is the highest scoring person not to solve question 1 and, as aforementioned, completed the very unusual feat of solving question three and obtaining a bronze medal, William's scores can be ordered so that they are in arithmetic progression, and Freddie came 2nd in the entire competition (mentioned twice because this is a rather impressive achievement!).

We are also able to construct a degree 5 polynomial $f : \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \to \mathbb{Z}$, where $f(x) =$ Score of UNK x (thus ensuring that our report is already superior to [4], since this is something that it failed to achieve on page 17):

$$f(x) = -\frac{19x^5 - 365x^4 + 2690x^3 - 9280x^2 + 14316x - 7980}{30}$$

This has the interesting property that the denominator is 30, and this was the 30th Balkan Mathematical Olympiad. Geoff and Gerry were wise not to sit the paper, since they would have had a tough time: $f(7) = -105$ and $f(8) = -582$. The reserve, Andrew Carlotti, however, would have put in a stellar performance, since $f(0) = 266$.

Finally, we came 5th in the team competition, which is the best ever UK result!

Events

The rest of this report is written in a diary format, detailing the events of each day. Solutions to questions are found in the appendix.

Friday 28th June

After successfully locating each other at London Heathrow airport (despite concerns that William may have forgotten to stop continuously travelling between Edinburgh and London at the wrong end), we acquired team t-shirts, UK badges and flags from Geoff and Gerry and headed towards the plane, with Geoff and Gerry providing our first mathematical problems of the trip. From Gerry, a question concerning travelators and shoelace tying: is it more efficient to tie your shoelaces before the travelator, or once you are on it? Clearly, by considering you and your double, it is always better to tie shoelaces on the travelator, so long as you don't mind irritating all the people trying to get past, and assuming that you are quick enough tying your shoelaces so that you don't fall off the end of the travelator. Geoff's problems concerns a North American luggage restrictions:

1. In North America, the maximum luggage size is defined in terms of the perimeter of the (cuboidal) piece of luggage. Suppose you are told that the perimeter of your case is too large. Is it possible that you could enclose the case within another one (orienting the case as necessary) so that the perimeter of the outside case is within the limits?

After we passed through security, we had some time to kill, so we lounged around in one of the seating areas. Frank realised that he hadn't brought an EU electrical adaptor with him. When Geoff gave him the option of using the shiny UKMT credit card to purchase a 'UKMT adaptor', he immediately shook off his torpor and his eyes lit up. He returned in double-quick time, clutching the sacred piece of kit. Undoubtedly this will eventually make its way to the Leeds headquarters, where it will be added to a tottering stockpile of paraphernalia for international maths competitions. During the course of our five-day stay in Cyprus, the UKMT will meet some more of our petty expenses, but it seems that, as a group, we are rather reluctant to purloin and squander large sums of money from the UKMT coffers.

The flight was uneventful, and we arrived in Larnaca airport at 9.30pm as expected. We then found the Macedonian team, who we shared a bus with to the Rodon hotel, where we would be staying. It was at this point that Geoff's international legendary status became apparent: one of the Macedonians appeared to be quite star-struck in his presence, recognising him immediately and declaring that he was a regular worshipper of Geoff's Facebook page whilst he struggled to resist the temptation to bow down in supplication before him[4]. The absolute power[5] of Geoff is clearly something that Adam P. Goucher can only hope to replicate.

We left Geoff with the Macedonian leader at a rather luxurious hotel in Nicosia, before journeying on towards a remote village in the mountains where we would be staying, namely Agros. Upon arrival at the hotel, the UK team disembarked from the bus with our cases. The Macedonians attempted to replicate this feat, but failed miserably, with two of them leaving their cases on the bus, which promptly drove off. Luckily, one of the cases had been picked up by a member of the UK team, but the other was still on the bus. Gerry made a *delightful* comment on the intelligence level of the Macedonian team, whilst the Macedonian deputy explained the situation to the hotel staff. By this point, it was 11.30pm and we were very hungry as we hadn't eaten anything since around 3.00pm, when we had a meal on the plane. There didn't appear to be any chance of

[4]All of the British students who have fallen under his spell at maths camps are aware of his god-like qualities, and some of us actually believe that he is God (with a capital G). He features prominently in the role plays that we enact during some break times in the camps, and he possesses many mystical powers (which we're not at liberty to reveal). He has appeared under various incarnations, such as a benevolent Santa Claus figure who distributes copies of UKMT publications to every boy and girl in the land at Christmas time, and a guard who will only let competent geometers through the Pearly Gates of Heaven.

[5]$|x^y|$

dinner, since the restaurant closed at 10.00pm and the hotel staff were trying to sort out the luggage situation. However, a phone call to the organiser of the competition from a disgruntled Gerry ensured that we weren't starved: we were soon provided with bread, cheese, a selection of unknown meats, tomato and cucumber. The UNKs were extremely grateful for this extra sustenance. Soon, Gerry was able to connect to the hotel Wi-Fi network (which proved to be rather unreliable during our stay). He gave us some updates on events at Wimbledon and also sounded a cautionary note by telling us the dangers of jumping off hotel balconies. This is quite pertinent, as all of our rooms had covered verandas. He lamented the lethal combination that arises when young people encounter alcohol [2, 8] and warned us not to attempt similar high jinks.

We were also given Balkan MO goodies, including caps, bags, tourguides, posters, t-shirts and stationery. Soon after this, we retired to bed, grateful for the air-conditioning in our rooms.

Saturday 29th June

In the morning, following Maria's acquisition of a Cypriot roommate who disagreed with her opinion that the temperature of the air conditioning should be set as low as possible, we attended the opening ceremony. Unfortunately, we were unable to see much of the ceremony, since we were seated near the back, and we couldn't hear much either due to the general murmurs from teams around us. There was also a highly annoying cameraman who felt it necessary to take photos at intervals of two milliseconds for the duration of the ceremony and who completely obstructed our view down the central aisle at times.

Features of the ceremony included a play 'Harmony of the spheres', and dancing by local students. The role play was unintelligible to many of us, not least because the action on the stage and the PowerPoint presentation that contained the English translations were often badly out of sync. We also encountered another mathematical problem, concerning whether we could all fit into the same room for the exam if there were desks as well - even without desks, the chairs were packed fairly close together, so we were uncertain what would happen. In essence, we were required to solve the problem:

2. How do you fit something bigger than a room into a room?

During the afternoon, we found some problems to do, the majority of which were inequalities or functional equations, such as the Balkan MO problems:

3. Does there exist an infinite sequence, $a_1, a_2, a_3, \dots$, such that $\sum_{i=1}^{n} a_i \leq n^2$ for all $n \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \frac{1}{a_i} \leq 2008$?

4. Find all functions $\mathbb{R} \to \mathbb{R}$ such that $f(f(x)+y) = f(f(x)-y) + 4f(x)y$.

We also had two visitors to the UK sofas. Firstly, a Romanian, whose shorts were included in the set of things obscured by his t-shirt, and who claimed he could solve the above functional equation, which we had been struggling with for a while, in ten minutes. He proceeded not to solve it. In fact, we believe his main purpose was espionage of the British TST problems. Soon afterwards, MKD 4, Antonij Mijoski joined us, giving us some more inequalities and a nice functional equation:

5. Find all functions $\mathbb{R} \to \mathbb{R}$ such that $xf(y) - yf(x) = f\left(\frac{y}{x}\right)$.

6. For positive reals x, y and z such that $x + y + z = \sqrt{2}$, prove that:

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \geq 2 + \frac{1}{\sqrt{3}}.$$

7. For reals a, b, and c, such that $a^4 + b^4 + c^4 = 3$, prove that:

$$\frac{9}{a^2 + b^4 + c^6} + \frac{9}{b^2 + c^4 + a^6} + \frac{9}{c^2 + a^4 + b^6} \leq 6 + a^6 + b^6 + c^6.$$

We moved onto discussing UK universities, and came to a unanimous (if we ignore UNKs with too few X chromosomes to have an opinion that actually matters) conclusion - first conjectured by the Macedonian - that Oxford is better than Cambridge. Adam P. Goucher would have been greatly displeased with Antonij, had he been invoking his omnipresence at that point in time.

The composite UNKs were abandoned as they solved more problems whilst the remainder of us woke up Frank to go for a walk into the village of Agros. We came across various buildings, including a church, a sports centre, and a supermarket. On the way back, we inevitably ended up discussing the cult status of Adam P. Goucher in UKMT activities. We determined that he would make an excellent local organiser at the Trinity camp, where he could continue refining Adam P. Goucher v2.0 (otherwise known as Gabriel Gendler). Agreed?

Dinner was served in the restaurant at 8.00pm. The hotel food was rather good on the whole, but it was sometimes difficult to tell exactly what we were eating. Gerry identified some pieces of spit-roasted chicken meat, none of which, it seemed, originated from a single animal. Warren and Oliver bravely sampled some of the hotel's fluorescent (and possibly radioactive) jellies. Contrary to expectations, the taste was rather bland, and we suffered no ill effects (although we probably shovelled down large quantities of E-numbers). We all agreed that, as a group, we were extremely unfussy eaters in comparison to members of other recent UK maths teams (see other student reports).

After dinner, Gerry explained to us some of the issues with secondary school mathematics and how teachers could extend their students' learning without leaving the syllabus. Examples included explaining irrational and rational numbers in more depth, proving that $\sqrt{2}$ is irrational, and showing why the proof doesn't work for $\sqrt{4}$, as well as proving the existence of rational numbers of the form a^b, where both a and b are irrational. He explained a particularly elegant proof, which shows the irrationality of $\sqrt{2}$:

Suppose $\sqrt{2}$ is rational. Then it can be written in the form $\frac{x}{y}$ with x and y coprime. Thus, any number of the form $a\sqrt{2} + b$, with a and b integers can be written in the form $\frac{ax+by}{y}$. As such, any positive number of this form is necessarily greater than $\frac{1}{y}$. Now, we know that $0 < \sqrt{2} - 1 < 1$, hence if we raise it to a high enough power, we can make it smaller than a fixed $\epsilon > 0$. Critically, we can make it smaller than $\frac{1}{y}$. But it can still be written in the form $\frac{ax+by}{y}$, since if we multiply out $(n-1)^m$, we always have integer coefficients. Hence, we have a contradiction, so $\sqrt{2}$ is irrational.

Gerry then went on to say how most A-level students don't understand why things go wrong when you try to integrate over a discontinuity. When asked about this topic, the stock-in-trade response from most students is 'you're not allowed to do it' (that's a fat lot of good to any serious mathematician). He provided us with a simple but illuminating example which he has used in the classroom before, claiming 'most people in the class don't get it, but there are some annoying people like James Aaronson who get it straight away':

8. Calculate

$$\int_{-1}^{1} \frac{1}{x} dx.$$

He also gave us the following problem, which was recognised from Gabriel Gendler's report of last year [4]:

9. What is the error in the following method of finding $n \in \mathbb{Z}$ such that $2^n < \frac{1}{2}$?

$$\begin{aligned} 2^n < \frac{1}{2} &\Rightarrow \log_{\frac{1}{4}}(2^n) < \log_{\frac{1}{4}} \frac{1}{2} \\ &\Rightarrow n \log_{\frac{1}{4}} 2 < \log_{\frac{1}{4}} \frac{1}{2} \\ &\Rightarrow n > \frac{\log_{\frac{1}{4}} \frac{1}{2}}{\log_{\frac{1}{4}} 2} \\ &\Rightarrow n > -1. \end{aligned}$$

Sunday 30th June

This was the day of the paper. Fifteen minutes before the exam started, we finally discovered the solution to the very important packing problem (question 2 in this report) which had been nagging at the back of our minds for the past 24 hours. It was very reassuring to see Geoff walk up the central aisle along with the other members of the jury, who had spent much of the last 48 hours in a conclave in the luxurious environs of the Nicosia Hilton hotel. A thumbs-up from the great man himself was definitely a good omen.

Unusually, the paper was sat from 10.00am to 2.30pm, which meant that we were able to have a lie in. It also meant that there was time for the acquisition of a second Cypriot roommate. Even more unusually, we were not allowed to hand in rough work, which seemed absolutely bizarre, and led to some heated discussions between invigilators and students afterwards when the students wanted to hand in their rough work. Some teams got around this by writing only on the white paper, which was intended for neat solutions only. The problems were:

10. In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, and the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral MKLP is cyclic.

11. Determine all positive integers x, y and z such that:

$$x^5 + 4^y = 2013^z.$$

12. Let S be the set of positive real numbers. Find all functions $f : S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:
(a) $xf(x, y, z) = zf(z, y, x)$,
(b) $f(x, yk, k^2z) = kf(x, y, z)$,
(c) $f(1, k, k + 1) = k + 1$.

13. In a mathematical competition, some competitors are friends; friendship is mutual, that is to say that when A is a friend of B, then B is also a friend of A. We say that $n \geq 3$ different competitors $A_1, A_2, \ldots, A_n$ form a weakly-friendly cycle if A_i is not a friend of A_{i+1} for $1 \leq i \leq n(A_{n+1} = A_1)$, and there are no other pairs of non-friends among the components of the cycle.
The following property is satisfied:

'for every competitor C and every weakly-friendly cycle S of competitors not including C, the set of competitors D in S which are not friends of C has at most one element.'

Prove that all competitors of this mathematical competition can be arranged into three rooms, such that every two competitors in the same room are friends.

Notable points include that no one solved question 4 (number 13 here) completely in the exam - the highest score obtained was 6. Also, a theorem of James Cranch, unknown to Warren, turned out to be incredibly useful. We shall state the theorem in its full form, though only part i) of it is necessary to answer question two (11 here). Proof of uniqueness of factorisation is left as an exercise to the reader.

Theorem: 2013, 2014 and 2015 are three consecutive numbers all with three distinct prime factors. Specifically, we have:
i) $2013 = 3 \times 11 \times 61$
ii) $2014 = 2 \times 19 \times 53$
iii) $2015 = 5 \times 13 \times 31$

Though part i) was the only part necessary to solve question two, we recommend that all students in year 11 and below learn ii) and iii) off by heart as well. Luckily, James Cranch is a great mathematician, therefore this theorem is well-known and quotable in olympiad exams. Incidentally, for those who are unaware, question two is an example of a Diophantine equation. The reason these equations are challenging and therefore often appear on olympiad exams is explained in [5]:'There is no algorithm for solving a generic Diophantine equation, which is why they can be very difficult to solve. [...] Even proving the non-existence of positive integer solutions to the innocuous-looking equation $x^n + y^n = z^n,\ n \geq 3$ (Fermat's last theorem) occupied mathematicians for three centuries before finally being settled by Andrew Wiles.'

When we came out of the exam, we were reunited with Geoff, who would now also be staying in the inferior-yet-still-superior-to-the-EGMO-youth-hostel hotel. We then had a late lunch, during which Geoff described the hotel's meat products (and Cypriot cuisine in general) as 'bony' and was understandably reluctant to sample some of the food (he decided to eat his meal at a more leisurely pace, and he waited until Gerry had tried the desserts. Gerry's various facial expressions helped him inform his choice of dessert.) Geoff's cunning strategy was inspired by his infallible method for cooking the perfect piece of toast:

Toasting the perfect piece of toast - *Geoff Smith*
Step 1: Acquire two pieces of bread, and a toaster.
Step 2: Place one piece of bread in the toaster and start the toaster.
Step 3: After 20 seconds, add the second piece of bread to the toaster.
Step 4: When the first piece of toast is burnt, stop the toaster and remove the second piece, which will be perfectly toasted.

Following lunch, Frank went back to sleep and we returned to the UK sofas to continue doing maths. We solved many elegant problems from Engel's problem solving book [3], for example:

14. Can you cut a thin hole into a plane, which leaves it connected, so that a wire model of (a) a cube of edge 1 (b) a tetrahedron of edge 1 can be pushed through the hole? The hole must have negligible area, and the thickness of the wires must be negligible.

15. Find infinitely many composite integers n such that $n | 2^{n-1} - 3^{n-1}$.

16. A curve C partitions the area of a parallelogram into two equal parts. Prove that there exist two points A, B of C such that the line AB passes through the centre O of the parallelogram.

Problem-solving can be an extremely strenuous activity, and so after slaving away at these problems along with many others all afternoon, we opted to buy the first of several 'UKMT ice-creams', before guaranteeing Warren's four IMO gold medals by ensuring that he knew the prime factorisations of all the years from 2013 to 2016. The first three UNKs decided to play tennis whilst the rest of us continued with the maths. The composite UNKs, despite having more factors than the other UNKs, appeared to have fewer interests, and continued doing maths until 2.30am the following morning, so we estimate that they completed at least fourteen hours of maths that day. To give an idea of how much this is, it is approximately the same amount of time as Frank spends sleeping every day.

Monday 1st July

This was essentially a free day for us, so naturally we spent the majority of it doing maths, whilst Gerry and Geoff, after staying up until midnight to finish marking our scripts, had to coordinate our marks. The coordination schedule was pinned up on the noticeboard in the foyer of the hotel, and after each coordination slot, we would nervously go downstairs to get confirmation of our marks from Geoff and Gerry. This year, coordination of the British scripts seemed to be a relatively straightforward process, and there were very few controversial scripts. Contrary to expectations, the problem 1 coordinators raised few objections to Frank's convoluted script, and this was duly awarded 10 points. Maria's wonderfully clean and short solution to problem 2 apparently bamboozled the coordinators, who thought at one point that it was flawed (because it was too simple). Due to the relative lack of progress on question 4, many people were awarded marks for recasting the problem in terms of graph theory and for starting an inductive proof.

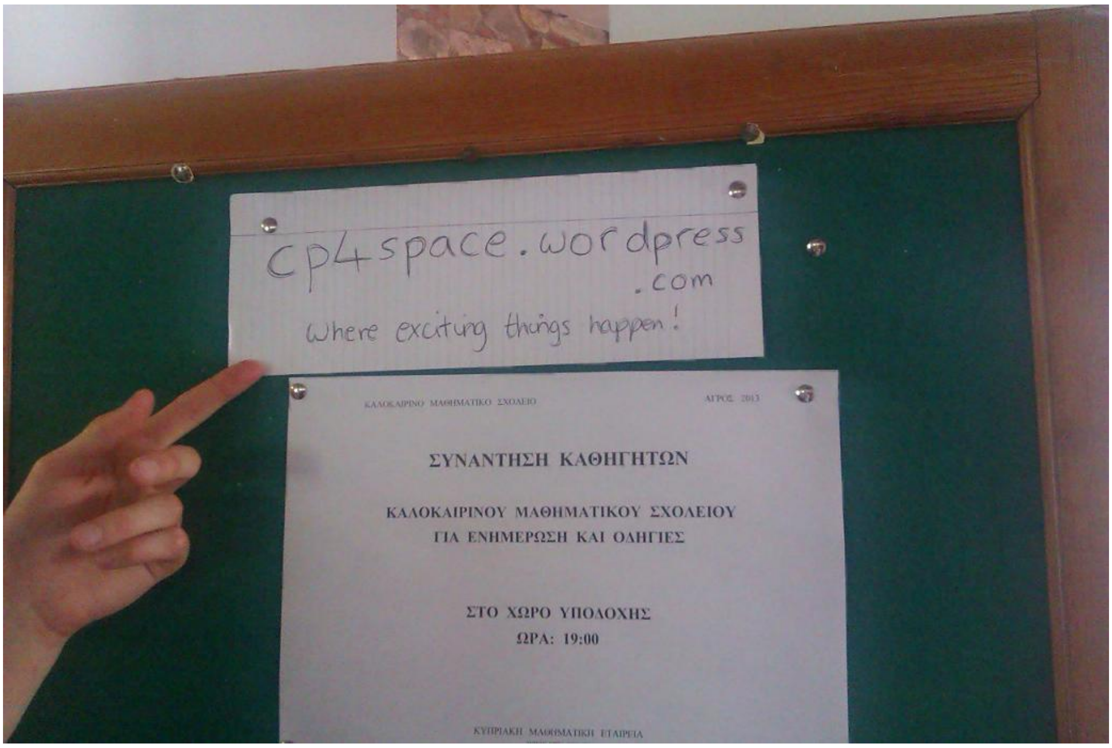
At lunchtime, we badgered Gerry for some money to buy extra bottled water (the location of this commodity in the hotel is apparently a closely guarded secret). Geoff egged us on furiously, telling us to take advantage of any opportunity to extract further riches from Gerry's horde. Gerry finally relented.

Once we all knew our final scores, we once again woke up Frank and abandoned the composite UNKs, who entertained themselves with the IMO 2010 shortlist problems, in order to go on another 'walk'. Supposedly, this was a nice gentle walk designed for the average tourist, including a nature trail which went 'up and down a bit', according to the hotel receptionist, who provided us with a terribly inaccurate map. In reality, it was a treacherous path of loose stones, steep hills, no signs (aside from the occasional one informing us of the name of a plant) and snakes, according to Frank, who we discovered is, quite frankly, scared of snakes. The other three UNKs didn't see the beast, and we mocked him mercilessly. After this episode, the hilarity mounted, and Frank continued to rail about the lurking dangers on this route. His primary concern was the likely presence of colonies of all manner of dangerous creatures such as tarantulas, snakes and lizards, and he was also worried that one of us might tumble down the hillside into the valley below. We explained that the hotel staff might not be as sadistic as he thought they were, but the tension was heightened, and every time Frank lost his footing, a strangled cry issued from behind.

The useless map also ensured that we got hopelessly lost, and were intending at one point to follow a road around to a different village, thinking that it was Agros. Luckily, the road in fact turned the other way, so led us away from where we were trying to get to and back to our hotel. We arrived back in time to find out the medal boundaries, which, unusually, meant that honourable mentions were impossible. There was also an interesting subplot. UNK 1 and MKD 1 (from Macedonia) achieved exactly the same score in the competition (20). UNK 1 was one of the UK team members who helped to rescue MKD 1's bag from the bus on the first day, and MKD 1 reciprocated this act of generosity. MKD 1 was the only official Balkan contestant to score 20 in this competition, and the silver medal boundary may well have been 21 had he not done so. UNK 1 wishes to place on record his thanks to MKD 1 for bringing down the silver medal boundary.

In the evening, we gathered around the TV to watch Andy Murray's fourth round Wimbledon match against Mikhail Youzhny. The match was tense, and there were some moments of real anxiety in the second set when the British number one lost four games in a row, but some spectacular shot-making sent him on his way to a stirring comeback. Two hours after we first turned on the television, match point arrived, and we were able to go down to dinner at 8:15 local time after the match finally came to a conclusion.

After dinner, we attempted to enrich the mathematical education of other teams by advertising the prestigious blog, cp4space, at the top of the notice board. It transpires that at least one member of the Cyprus team had already heard of it, so it's entirely possible that the advert is superfluous and that cp4space is already known worldwide.



We also rediscovered G4 from the 2010 IMO shortlist, a problem which can (much to Freddie's distaste) be bashed with areals. The Euclidean solution which we found, however, is far nicer, and as such, the problem and its solution have made this report:

17. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc BDC and F a point on the side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

Tuesday 2nd July

We awoke earlier than usual in order to take part in the excursion, which involved a long coach journey to Nicosia and back. Whilst in Nicosia, we visited a museum which contained various items of pottery, some statues, and a few tombs from ancient Cyprus, as well as a film about a glazing experiment, during which Frank fell asleep. We then purchased more UKMT ice creams and Freddie and Warren continued with the IMO 2010 shortlist (now on C5) before visiting the university of Cyprus, which we were encouraged to apply to. It was hoped that speeches from the Rector of the university, the head of the Mathematics Department and the admissions secretary would whet our appetite for the courses offered at the university. Unfortunately, the courses were taught in Greek, which could be slightly problematic for our team. We ended up at the end of a very long lunch queue, which led to concerns about the capacity of the cafeteria, given that there were only 100 of us and nearly 7000 students. There was also a rather curious decoration on the ceiling, consisting of wooden strips arranged in a chessboard fashion, and indented at what appeared to be random intervals.

We enjoyed a meal which Geoff described as ‘adequate’, and some of the UNKs, including UNK7, were keen to purchase yet more UKMT ice creams, which they guzzled enthusiastically. Afterwards, Geoff shared some anecdotes about previous IMO immigration queues and luxury flights from Saudi Arabia. We also discovered that the statement and the solution of the Serbian problem 4 in the contest had already appeared in one of the Serbian leader’s recent papers [1]. Geoff and Gerry tried to remember his name, and they debated whether or not an extra vowel is present in his surname (Gerry is correct on this occasion). To prevent future occurrences of this catastrophic event, we suggest to the Serbian leader that he becomes a middle initialismist in order to make his name more memorable (c.f. Adam P. Goucher).

Upon returning to the hotel, we watched tennis and Maria took part in the international cipher solving contest of cp4space. We then donned our UK flags (the right way up), team t-shirts and badges, and made our way to the closing ceremony, where, after a few short speeches, we received our prizes. Geoff was invited by the host to move a small vote of thanks on behalf of the guest countries (his name was somehow mispronounced so that it sounded more like ‘John Smith’). He gave an overwhelming speech which almost brought everyone in the room to tears, explaining that the accomplishments of the Cypriot organisers seemed to defy Parkinson’s universal law (work expands so as to fill the time available for its completion). ‘Work is compressible!’ he declared to the room at large.

The UNK medal winners were then presented with certificates and pieces of shiny metal by the Minister for Education and Culture and the Chairman of the Jury, Dr Gregory Makrides. Gerry valiantly (and often successfully) attempted to photograph us on the stage, despite the best efforts of other deputy leaders to get in his way.

There was then a short period of everyone excitedly taking photos. Antonij's lifelong dream was also realised when he had a photo taken of him with his idol (Geoff). Geoff explained that ever since his IMO 2011 Windmill problem took the mathematical olympiad community by storm (reactions ranged from 'The best combinatorics IMO problem, the best IMO problem 2, the best IMO style IMO problem, I've met' to 'F*** the windmills' [6]), he has been in great demand during photo opportunities, when many excitable young mathematicians clamour to get photos with famous olympiad icons.

We then proceeded downstairs to an outdoor area where we ate our final meal at the hotel, which featured folk dancing and singing for entertainment. Here, it transpired that none of the UNKs has dipped so much as a single toe into one of the hotel's swimming pools. Although we missed out on this unique outdoor swimming experience, most of us derive more pleasure and satisfaction from solving hard maths problems.

Upon returning upstairs, we acquired our scripts, presented Geoff with his diploma for being leader, and admired Frank's convoluted solution to question one which included plenty of extra points and isosceles triangles for good measure. Finally, we went to sleep, ready for an early departure in the morning.

Wednesday 3rd July

Leaving the hotel at 6.30am, we returned on a coach to Larnaca airport where we had breakfast before boarding the plane. Oliver successfully got a pair of compasses through security, despite them being detected, leading to a bag search. On the plane, we solved an inequality which had stumped the Greek deputy leader for three days:

18. Given reals, a, b and c such that $a + b + c = 0$ and $c \geq 1$, prove that:

$$a^4 + b^4 + c^4 - 3abc \geq \tfrac{3}{8}$$

We also wrote the report, wrote that we were writing the report, wrote that we wrote that we were writing the report... When we returned to Heathrow following another uneventful flight, we parted ways and returned to our respective homes.

Thanks

Thanks go to Gerry and Geoff, for keeping us entertained with lots of problems and for generally being excellent leaders. It's highly likely that without Gerry we would have starved before even the second day. Also, thank you to the Cypriot organisers and the Cyprus Mathematical Society (in particular the Chairman, Dr Gregory Makrides and the Minister for Education and Culture, Mr Kyriakos Kenevezos) for organising such a great event at such short notice, without this having any effect on the competition. We are also grateful to the helpful staff at the Rodon hotel, who showed us excellent hospitality, and the teams who were such great company for the entire trip. Finally, we must sincerely thank Adam P. Goucher for having such an inflated ego, so that we can mention him numerous times without him minding, and for the effort on his part for progressing from a figment of Geoff's fevered imagination into an extremely responsible figure whom we all admire.

Disclaimer

This report is written from an unbiased perspective. Commendations of Geoff's speech are not an attempt to stay on his good side in order to improve chances of certain team members for becoming EGMO deputy leader at some point in the future. Furthermore, attempts and recommendations to persuade Geoff that Adam P. Goucher would make an excellent local organiser were certainly not ordered by Adam P. Goucher himself (not least because they would be unnecessary: it seems natural that Geoff would have invented someone only if they were capable of being a fantastic local organiser, hence any other conclusion is automatically wrong). Reading this report does affect your statutory rights.

Appendix

1. (Freddie) The answer is no. For a cuboid X with dimensions a, b, c, denote the surface area, perimeter and longest diagonal as SA_X, P_X, and D_X respectively. Note that we have:

$$P_X = 4(a+b+c), SA_X = 2(ab+bc+ca), D_X^2 = a^2+b^2+c^2.$$

Now,

$$P_X^2 = 16(a+b+c)^2 = 16(a^2+b^2+c^2+2ab+2bc+2ca) = 16(D_X^2+SA_X). \quad (1)$$

Denote the larger cuboid as L and the smaller cuboid as S. By (1), it suffices to show $D_L \geq D_S$ and $SA_L \geq SA_S$.
The former follows from L containing S and the latter is shown as follows: consider a rectangular face F_i of S, and imagine projecting F_i onto L orthogonally from S, producing F_i^*. Now, the area of of F_i^* is greater than or equal to the area of F_i, and no two F_i^*s overlap, thus, as required,

$$SA_S = \sum F_i \leq \sum F_i^* \leq SA_L.$$

2. Unknown to us, the room in fact had a partition wall which could be folded up, thus increasing the volume of the room a sufficient amount so that all the desks and chairs could fit into the room. Hence, the only solution to the problem is to extend the room.

3. (Freddie) The answer is no. We have:

$$\sum_{i=k+1}^{2k} a_i \leq \sum_{i=1}^{2k} a_i \leq 4k^2.$$

By Cauchy-Schwarz:

$$(\sum_{i=k+1}^{2k} a_i)(\sum_{i=k+1}^{2k} \frac{1}{a_i}) \geq k^2 \Rightarrow \sum_{i=k+1}^{2k} \frac{1}{a_i} \geq \frac{k^2}{\sum_{i=k+1}^{2k} a_i} \geq \frac{k^2}{4k^2} = \frac{1}{4}.$$

Hence $\sum_{i=1}^{\infty} a_i$ is unbounded, so cannot be less than 2008.

4. Let $g(x) = f(x) - x^2$. Then we have

$$g(g(x)+x^2+y) = g(g(x)+x^2-y).$$

Setting $y = -g(a) - a^2 + b$, we obtain

$$g(g(x)+x^2-g(a)-a^2+b) = g(g(x)+x^2+g(a)+a^2-b) = g(g(a)+a^2-g(x)-x^2+b)$$

(by symmetry). Setting $b = g(a)+a^2-g(x)-x^2+c$, this becomes

$$g(c) = g(2(g(a)+a^2-g(x)-x^2)+c).$$

If $g(x)+x^2$ is constant, we have $f(x) = k$ for k a constant. It is easy to verify that the only solution of this form is $f(x) = 0$. Otherwise, for some a, x, we have $g(a)+a^2-g(x)-x^2 = t \Rightarrow g(c) = g(2t+c)$. Thus g is periodic, with period $2t$. Setting $a = x + 2t$ yields

$$\begin{aligned} g(c) &= g(2(g(x+2t) + (x+2t)^2 - g(x) - x^2) + c) \\ &= g(8(t^2 + xt) + c). \end{aligned}$$

As x ranges over the reals, $t^2 + xt$ takes any value. Thus, g is necessarily constant, and we obtain $g(x) = x^2 + k$ for a constant k. It is easy to check this works.

5. (Freddie) Let $P(x, y)$ denote the question statement. Then:

$$P(2,0) \Rightarrow 2f(0) = f(0) \Rightarrow f(0) = 0, \quad (1)$$

$$P(1,1) \Rightarrow f(1) - f(1) = f(1) \Rightarrow f(1) = 0, \quad (2)$$

and, from (2),

$$P(x,1) \Rightarrow xf(1) - f(x) = f\left(\frac{1}{x}\right) \Rightarrow f(x) = -f\left(\frac{1}{x}\right). \quad (3)$$

Now,

$$P\left(\frac{1}{y}, \frac{1}{x}\right) \Rightarrow \frac{1}{y} f\left(\frac{1}{x}\right) - \frac{1}{x} f\left(\frac{1}{y}\right) = f\left(\frac{y}{x}\right),$$

so from (3),

$$f\left(\frac{y}{x}\right) = \frac{1}{x} f(y) - \frac{1}{y} f(x).$$

But

$$f\left(\frac{y}{x}\right) = xf(y) - yf(x).$$

Hence

$$f(y)\left(x - \frac{1}{x}\right) = f(x)\left(y - \frac{1}{y}\right).$$

Therefore, we have

$$f(x \neq 0) = k\left(x - \frac{1}{x}\right), f(0) = 0, k \in \mathbb{R}$$

It is easy to verify that this solution works.

6. We prove a lemma, from which the result follows immediately:

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} \geq \frac{1}{\sqrt{1+(a+b)^2}} + 1$$

where $a + b \leq \sqrt{2}$.
Proof: First note that by AM-GM, $ab \leq \frac{1}{2}$. Squaring both sides, we require

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{2}{\sqrt{(1+a^2)(1+b^2)}} \geq 1 + \frac{1}{1+(a+b)^2} + \frac{2}{\sqrt{1+(a+b)^2}}.$$

We first prove that

$$\frac{2}{\sqrt{(1+a^2)(1+b^2)}} \geq \frac{2}{\sqrt{1+(a+b)^2}}.$$

Squaring both sides, multiplying through by denominators and cancelling, we require $2 \geq ab$, which is true. Now we only need

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} \geq 1 + \frac{1}{1+(a+b)^2}.$$

Once again, multiplying through by denominators and cancelling gives that we require

$$2ab \geq 2a^2b^2 + a^4b^2 + a^2b^4 + 2a^3b^3,$$

or equivalently,

$$2 \geq ab(2+(a+b)^2).$$

But we are done because

$$ab(2+(a+b)^2) \leq \frac{1}{2}(2+2) = 2.$$

7. (Warren) Firstly, we show that $a^2+b^2+c^2 \leq 3$. This is because, by the Cauchy-Schwarz inequality,

$$(a^2+b^2+c^2)^2 \leq (a^4+b^4+c^4)(1+1+1) = 9.$$

Next, from the Cauchy-Schwarz inequality we can see that

$$9 = (a^4+b^4+c^4)^2 \leq (a^2+b^4+c^6)(a^6+b^4+c^2)$$

so

$$\begin{aligned}
\sum_{cyc} \frac{9}{a^2+b^4+c^6} &\leq \sum_{cyc} \frac{(a^2+b^4+c^6)(a^6+b^4+c^2)}{a^2+b^4+c^6} \\
&= \sum_{cyc}(a^6+b^4+c^2) \\
&= \sum_{cyc} a^6 + \sum_{cyc} a^4 + \sum_{cyc} a^2 \\
&\leq \sum_{cyc} a^6 + 3 + 3 \\
&= 6 + a^6 + b^6 + c^6.
\end{aligned}$$

8. (Oliver) An incautious student might write

$$\int_{-1}^{1} \frac{1}{x} dx = [\ln|x|]_{-1}^{1} = 0.$$

(Gerry hates the notation $\ln|x|$; he thinks this should be banned from all schools.)
This might look OK, but a lot of 'fudging' has gone on here; the student hasn't realised that something might go wrong at $x = 0$. So what if you take the following approach (where ϵ is a small positive number)?

$$\int_{-1}^{1} \frac{1}{x} dx = \lim_{\epsilon \to 0} \left(\int_{-1}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{1} \frac{1}{x} dx \right) = \lim_{\epsilon \to 0} (0 - \ln \epsilon + \ln \epsilon - 0) = 0$$

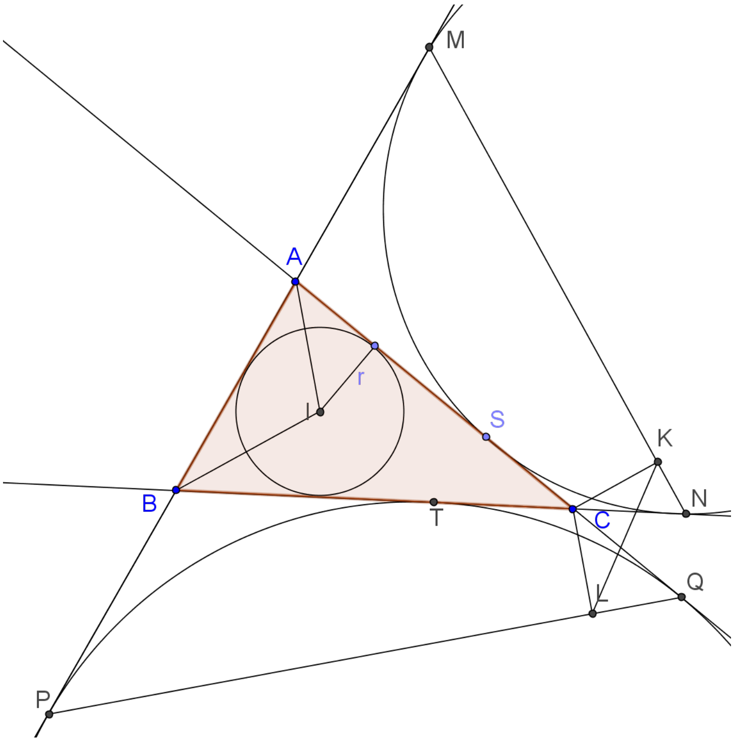
This looks much more sensible, and there seems to be a lot less fudging. You're using incremental values in order to get around the inconvenience of the discontinuity at $x = 0$. After all, aren't you employing a similar method when you differentiate from first principles? But there's something seriously wrong here. What is it? (The reader is encouraged to give this a moment's thought if they haven't seen this before.)
The nature of the limiting process is suspect. For example, why not write:

$$\int_{-1}^{1} \frac{1}{x} dx = \lim_{\epsilon \to 0} \left(\int_{-1}^{-2\epsilon} \frac{1}{x} dx + \int_{3\epsilon}^{1} \frac{1}{x} dx \right) = \lim_{\epsilon \to 0} (0 - \ln 2\epsilon + \ln 3\epsilon - 0) = \ln \frac{2}{3}.$$

As long as the negative and positive increments in the two integrals reach 0 'at the same time', you'd think it would be fine to carry out such a procedure. But it's obviously not okay, as you actually can get any answer you like. Gerry gave us his door-closing analogy: the above limiting process could be compared to a situation where a weak person tries to push open a door that a stronger person is trying to close.
He says that this example could lend itself to a simple discussion of integrability of functions for secondary school students. There exist functions which are not actually integrable over a certain interval, but if you partition them in a certain way, you might (wrongly) conclude that they are integrable over that interval.

9. As described in Gabriel's report, the first step, taking logs base $\frac{1}{4}$, is incorrect, since $log_{\frac{1}{4}} x$ is a decreasing function, hence the inequality sign should be reversed at this point. In school, we are wrongly taught that you can treat an inequality in much the same way as you treat an equality, with the condition that if you multiply by -1, the sign must be reversed.

10.



Let the incentre of triangle ABC be I. We prove that triangles AIB and KCL are similar. We have:

$$\begin{aligned}
\angle BIA &= 180^\circ - \angle IAB - \angle ABI && \text{(angles in a triangle)}\\
&= 180^\circ - \frac{1}{2}\angle CAB - \frac{1}{2}\angle ABC && \text{(I is where the angle bisectors meet)}\\
&= \angle BCA + \angle CAB + \angle ABC - \frac{1}{2}\angle CAB - \frac{1}{2}\angle ABC && \text{(angles in a triangle)}\\
&= \angle NCQ + \frac{1}{2}\angle CAB + \frac{1}{2}\angle ABC && \text{(vertically opposite angles)}\\
&= \angle NCQ + \frac{1}{2}(180^\circ - \angle APQ - \angle PQA) + \frac{1}{2}(180^\circ - \angle BNM - \angle NMB)\\
&= \angle NCQ + (90^\circ - \angle PQA) + (90^\circ - \angle BMN)\\
& && \text{(triangles } APQ \text{ and } BMN \text{ are isosceles by equal tangents)}\\
&= \angle NCQ + \angle QCL + \angle KCN\\
& && \text{(} K \text{ and } L \text{ are projections from C, so triangles } KCN \text{ and } CLQ \text{ are right-angled)}\\
&= \angle KCL.
\end{aligned}$$

Also,

$$\frac{AI}{BI} = \frac{s-a}{\sin(90 - \frac{\angle CAB}{2})} \div \frac{s-b}{\sin(90 - \frac{\angle ABC}{2})}$$

$$= \frac{(s-a)\sin(90-\frac{\angle ABC}{2})}{(s-b)\sin(90-\frac{\angle CAB}{2})}.$$

Let S be where ω_b touches AC, and T be where ω_a touches BC. Then by equal tangents, $CT = CQ$ and $CS = CN$. It is well known that $CT = s-b$ and $CS = s-a$, hence

$$CK = CN\sin(\angle CNK) \Rightarrow CK = (s-a)\sin(90-\frac{\angle ABC}{2}),$$

$$CL = CQ\sin(\angle LQC) \Rightarrow CL = (s-b)\sin(90-\frac{\angle CAB}{2}),$$

$$\therefore \frac{CK}{CL} = \frac{(s-a)\sin(90-\frac{\angle ABC}{2})}{(s-b)\sin(90-\frac{\angle CAB}{2})} = \frac{AI}{BI}.$$

Thus triangles AIB and KCL are similar, as required. Now,

$$\begin{aligned}
\angle PLK &= 90+\angle CLK \\
&= 90+\angle ABI \\
&= 90+\angle\frac{ABC}{2} \\
&= 180-(90-\angle\frac{ABC}{2}) \\
&= 180-\angle NMB = 180-\angle KMP.
\end{aligned}$$

Therefore $MKLP$ is cyclic.

11. We present two proofs. One is for people who dislike number theory and would rather convert everything to algebra. It is extremely elegant.
Both arguments begin by showing that $5|y$: We know by Fermat's Little Theorem that $x^5 \equiv -1, 0, 1 (\text{mod } 11)$. Since the $2013 \equiv 0 (\text{mod } 11)$, we require that $4^y \equiv -1, 0, 1 (\text{mod } 11)$. Computing powers of 4 modulo 11, we see that the order of 4 modulo 11 is 5, hence we must have $5|y$. Hence, we may factor the LHS as

$$(x^5 + y^{5k}) = (x+4^k)(x^4 - 4^k x^3 + 4^{2k}x^2 - 4^{3k}x + 4^{4k}).$$

Furthermore, these factors are coprime, since

$$(x^4 - 4^k x^3 + 4^{2k}x^2 - 4^{3k}x + 4^{4k}) = (x+4^k)(x^4 - 2\times 4^k x^3 + 3\times 4^{2k}x^2 - 4\times 4^{3k}x) + 5\times 4^{4k}.$$

Any common divisor must divide 2013 and 5×4^{4k}. Since these are coprime, the greatest common divisor is 1.
The proofs split here. We first present a number theory method:

Note that $3 \nmid x$, since $3|2013$ but $3 \nmid 4^y$. By considering modulo 2 we can deduce also that x is odd. We consider the second factor modulo 3 and modulo 4:

$$\begin{aligned}(x^4 - 4^k x^3 + 4^{2k}x^2 - 4^{3k}x + 4^{4k}) &\equiv 1 - x + 1 - x + 1 \\ &\equiv -2x \pmod 3 \qquad (1). \\ (x^4 - 4^k x^3 + 4^{2k}x^2 - 4^{3k}x + 4^{4k}) &\equiv 1 \pmod 4 \qquad (2).\end{aligned}$$

Since $3 \nmid x$, we must have that $3|x + 4^k$. Hence, $x \equiv 2 \pmod 3$. There are now three possibilities for the second factor. It is either $11^z, 61^z$ or $(11 \times 61)^z$). We evaluate all these possibilites modulo 3 and 4, evaluating the parity of z by using from (1) and (2) that the second factor is congruent to 2 modulo 3 and 1 modulo 4.

Factor	modulo 3	z	modulo 4	z
11^z	2^z	Odd	3^z	Even
61^z	1^z	Impossible	-	-
$(61 \times 11)^z$	2^z	Odd	3^z	Even

Clearly, z cannot be both even and odd at the same time, hence none of these cases are possible, and there are no solutions to the equation.

The other proof uses the following inequality:

$$(x + 4^k)^5 \geq x^5 + 4^{5k} \geq \frac{(x + 4^k)^5}{16}$$

The LHS can be proven by simply expanding out terms, and the RHS follows from the power mean inequality. Next, suppose $x + 4^k = 3^z$. Then we must have:

$$(x + 4^k)^5 = 3^{5z} \geq x^5 + 4^{5k} = 2013^z \Rightarrow 243^z \geq 2013^z$$

This is clearly a contradiction. The other case is that $x + 4^k \geq 11^z$. This yields:

$$\frac{11^{5z}}{16} \leq \frac{(x + 4^k)^5}{16} \leq x^5 + 4^k = 2013^z \Rightarrow 11^{5z} \leq 16 \times 2013^z$$

This is also a contradiction. Hence, there are no solutions.

12. We have:

$$\begin{aligned}f(x, y, z) &= f(x, \sqrt{z}\frac{y}{\sqrt{z}}, z) \\ &= \sqrt{z} f(x, \frac{y}{\sqrt{z}}, 1) && \text{by (b)} \\ &= \frac{\sqrt{z}}{x} f(1, \frac{y}{\sqrt{z}}, x) && \text{by (a)} \\ &= \frac{\sqrt{z}}{xm} f(1, \frac{my}{\sqrt{z}}, m^2 x) && \text{by (b).}\end{aligned}$$

We choose m such that

$$\frac{my}{\sqrt{z}} = k,; \quad (1)$$

$$m^2x = k+1, \quad (2)$$

giving

$$f(x,y,z) \quad = \quad \frac{\sqrt{z}}{xm} \times m^2x = m\sqrt{z} \qquad \text{by (c).}$$

To determine the value of m in terms of x, y and z, we solve (1) and (2) simultaneously. This results in the quadratic equation:

$$xm^2 - \frac{ym}{\sqrt{z}} - 1 = 0 \Rightarrow m = \frac{y \pm \sqrt{y^2+4xz}}{2x\sqrt{z}}.$$

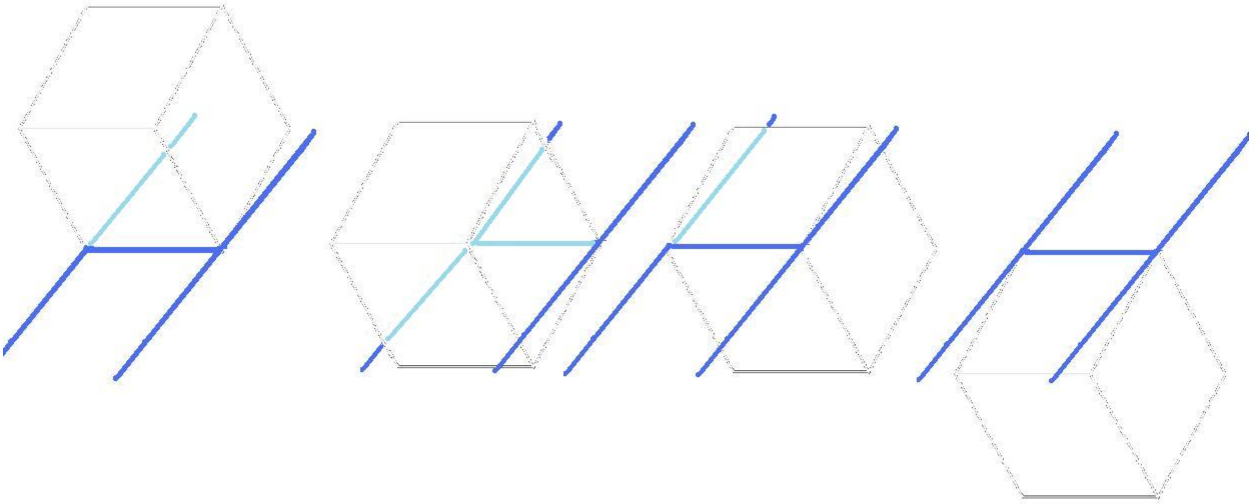
Since $f(x,y,z) > 0$ we require the positive root. Hence the only possible solution is

$$f(x,y,z) = \frac{y + \sqrt{y^2+4xz}}{2x}.$$

It is easy to verify that this works.

13. We convert the problem into an equivalent one using graph theory, taking the complement. The condition then becomes that for each vertex x not involved in a cycle of three or more competitors, x is connected by an edge to at most one vertex in the cycle. The problem is to show that the vertices can be partitioned into three sets, such that no vertex is connected to another in the same set. This is equivalent to showing such a graph is three-colourable, which can be shown by induction if there exists a vertex of degree at most two - the base case is trivial, so suppose the graph is three-colourable when there are fewer than n vertices. When there are n vertices, we take the one of degree at most two and remove it. The resulting graph is three-colourable by the induction hypothesis. We then add the vertex back in, giving it a colour different to the two vertices it is connected to. To prove there is a vertex of degree at most two, we consider the longest path P where no two vertices in the path are connected to each other and assume for contradiction that all vertices have degree at least three. Let one end of P be the vertex a. Then a must be connected to two vertices not in P, say b and c. Now, b and c must each be connected to at least one vertex in P, else P can be extended to include b or c, contradicting the maximality of P. Suppose b is connected to b_P in P and c is connected to c_P in P (with c_P, b_P not necessarily distinct). WLOG, assume b_P is at least as close to a than c_P is. Then b is connected to both a and b_P, which are both part of the cycle b_P, ..., c_P, c, a, ..., b_P. But this contradicts the problem statement, hence there must be a vertex with degree at most two..

14. The answer is yes to both questions. For the cube, it is possible to do so with a H shaped cut:



For the tetrahedron, a T shaped cut works. The reader is invited to work out how.

15. We prove that

$$3^{2^k} - 2^{2^k} | 3^{3^{2^k} - 2^{2^k} - 1} - 2^{3^{2^k} - 2^{2^k} - 1}.$$

Note that for $k > 0$, this is composite since we can factor it as the difference of squares, with both the factors greater than 1. Clearly, if $a|b$ we have that $x^a - y^a | x^b - y^b$, since we may write $b = ma$. When $x^a = y^a, x^{ma} = y^{ma}$, hence $(x^a - y^a)$ is a factor of the polynomial $x^{ma} - y^{ma}$. Now, we show by induction that

$$2^k | 3^{2^k} - 2^{2^k} - 1$$

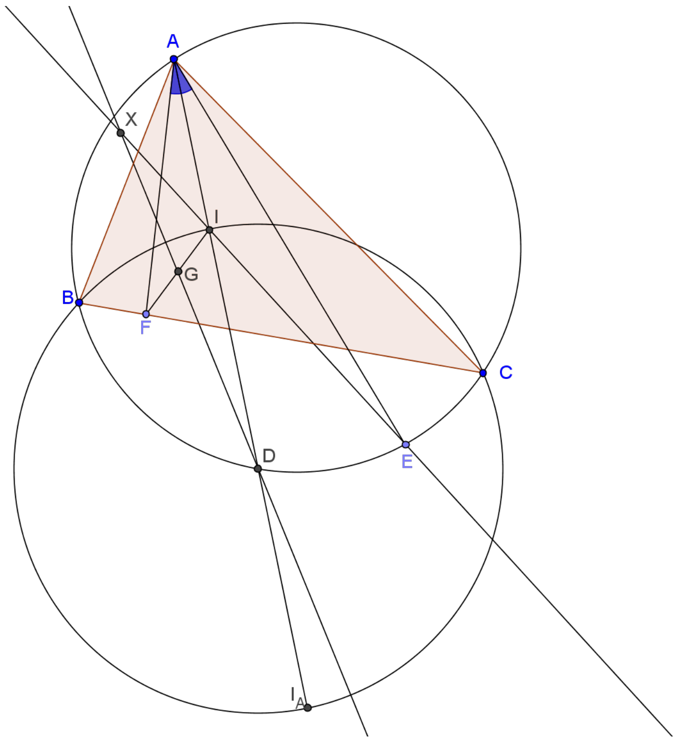
The base case, $k = 0$ is trivial. Assume for $k = n - 1$. For $k = n$, we have

$$3^{2^n} - 2^{2^n} - 1 = (3^{2^{n-1}} - 2^{2^{n-1}} - 1)(3^{2^{n-1}} + 2^{2^{n-1}} + 1) + 2^{2^n}.$$

Since, by the inductive hypothesis, the first parenthesised expression is divisible by 2^{n-1}, the second is even, and $2^n \geq n$, we have that $2^n | 3^{2^n} - 2^{2^n} - 1$ and we are done.

16. The case when C passes through O is trivial. So assume C does not pass through O. Then we can reflect the C through O to create C'. If C and C' do not intersect, the C cannot split the parallelogram into two parts, since C and C' split the parallelogram into three parts with non-zero area, two of which must be halves if C splits the parallelogram in half. This is clearly impossible. So the two curves intersect at B, say. Then the reflection of B in O (B', say) lies on both C and C' also. Hence BB' passes through O.

17.



(Freddie) Produce AIO to I_A, the excentre opposite A, nothing that D is the centre of circle $BICI_A$ which has diameter II_A (this is a well-known result; see Theorem 2.13 of [7] for details) and let $DC \cap EI = X$. We need to show that $X \in \Gamma$, i.e. X, A, D and E are concyclic.
In triangles AEC, ABF:

$$\angle FAB = \angle CAE,$$

Also, by angles in the same segment,

$$\angle ABF = \angle ABC = \angle AEC.$$

Therefore, triangles AEC and ABF are similar. Hence

$$\frac{AC}{AF} = \frac{AF}{AB} \Rightarrow AB \cdot AC = AE \cdot AF.$$

Also, it is well-known that $AI \cdot AI_A = AB \cdot AC$. Therefore,

$$AI \cdot AI_A = AE \cdot AF \Rightarrow \frac{AI}{AF} = \frac{AE}{AI_A}.$$

Also, note that $\angle IAF = \angle EAI$. Therefore, triangles AIE and AFI_A are similar. Now, G is the midpoint of IF, D is the midpoint of II_A and D is the centre of the circle on diameter II_A. So, by the midpoint theorem and corresponding angles:

$$DG || I_AF \Rightarrow \angle GDA = \angle FI_AA.$$

So, we have

$$\begin{aligned}\angle XDA &= \angle GDA \\ &= \angle FI_AA \\ &= \angle IEA \\ &= \angle XEA.\end{aligned}$$

Hence, by the converse of angles in the same segment, X, A, D and E are concyclic, as required.

18. First note that if one of $a, b \geq 0$ then the other is negative, so:

$$a^4 + b^4 + c^4 - 3abc \geq c^4 \geq 1,$$

so we may assume $a, b < 0$
Now,

$$a^3 + b^3 + c^3 - 3abc \equiv (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

So

$$a + b + c = 0 \Rightarrow 3abc = a^3 + b^3 + c^3.$$

So we need to show

$$(a^4 - a^3) + (b^4 - b^3) + (c^4 - c^3) \geq \frac{3}{8}.$$

Now, $c \geq 1$, so $c^4 \geq c^3$, which means we need to show for $a, b < 0$, $a + b \leq -1$ that $(a^4 - a^3) + (b^4 - b^3) \geq \frac{3}{8}$.
Let $f(x) = x^4 - x^3$. Then $f''(x) = 12x^2 - 6x = 6x(2x-1)$, which is positive for $x < 0$, so f is convex for $x < 0$. By Jensen:

$$\begin{aligned}f(a) + f(b) \geq 2f(\frac{a+b}{2}) \Rightarrow (a^4 - a^3) + (b^4 - b^3) &\geq 2(\frac{a+b}{2})^4 - \frac{a+b}{2})^3) \\ &\geq 2((-\frac{1}{2})^4 - (-\frac{1}{2})^3) \\ &= 2(\frac{1}{16} + \frac{1}{8}) \\ &= \frac{3}{8},\end{aligned}$$

with equality if and only if $a = b = -\frac{1}{2}, c = 1$.

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