25th BALKAN MATHEMATICAL OLYMPIAD

OHRID, MACEDONIA 4-10 May 2008

REPORT OF THE UK AND IRELAND TEAM

Written by UK and Ireland Team Leader (Adrian Sanders)

Most people involved with olympiad mathematics will be familiar with the International Mathematical Olympiad. The IMO is the most prestigious maths competition for school pupils. It is held annually in July, in a different country each year. The process for selecting the UK team for the IMO is a long one, beginning with the Senior Maths Challenge taken in November, and proceeding through the first and second rounds of the British Mathematical Olympiad. The twenty best performers in the second round of the BMO are invited to a training session at Trinity College, Cambridge in April, where they sit a further two selection exams to get even closer to making the cut for the IMO team of six, which is selected after a final training and selection camp at Oundle School in May.

It may be less well known to some British readers that there are several other international mathematics competitions for school pupils, mostly regionally based. Among the most venerable of these regional competitions is the Balkan Mathematical Olympiad, first held in 1984. Since *BMO* tends to mean something else for British readers, I'll call it the *BalkMO*. The Balkan Olympiad consists of a single 4½ paper of 4 problems, drawn from the topics of Olympiad mathematics. The UK has competed in the BalkMO as a guest nation since 2005. We were delighted to receive the invitation to do so again this year for the 25th BalkMO, held in Ohrid, Macedonia. For the first time we participated not as the UK, but as *UK and Ireland*, a combined team for the whole British Isles. This enabled us to include the excellent young Irish-based mathematician Galin Ganchev on the team.

The UK and Ireland team were selected on the results of the IMO selection exams taken at the Trinity Training Session in April 2008. Unlike most countries we chose not to take some of our very strongest competitors, opting instead for a team of students who missed out narrowly on the reckoning for the IMO team, mixed with talented younger students for whom the BalkMO could be a proving ground for IMO participation in coming years.

The UK and Ireland team for the 25th BalkMO was:

UNK & IRL 1	Galin Ganchev (Castletroy College, Ireland)
UNK & IRL 2	Andrew Hyer (Westminster School)
UNK & IRL 3	Peter Leach (Monkton Combe School)
UNK & IRL 4	Craig Newbold (Whitley Bay High School)
UNK & IRL 5	Hannah Roberts (School of St. Helen and St. Katherine, Abingdon)
UNK & IRL 6	Rong Zhou (Bristol Grammar School)

Other guest nations were invited too: Azerbaijan; France, for the first time; Italy, whose IMO performance has been so impressive in recent years; Kazakhstan; Macedonia B; Tajikistan; and Turkmenistan.

Along with the 11 full BalkMO participants (Albania, Bosnia and Herzegovina, Bulgaria, Cyprus, Greece, Macedonia, Moldova, Montenegro, Romania, Serbia and Turkey) this brought the total number of teams at BalkMO 2008 to 19.

The leader of the UK and Ireland team was Adrian Sanders, formerly of Trinity College, Cambridge. The deputy leader was Jacqui Lewis of St. Julian's International School, Carcavelos, Portugal.

The UK and Ireland team did well. Galin won a silver medal; Andrew, Peter, Craig and Hannah won bronze medals. In a league table of team performance based on total score, UK and Ireland placed 8th in the competition. Against the stiff competition of the Balkan countries (most of whom, unlike the UK and Ireland, bring their strongest team to this competition as well as the IMO) this was a creditable performance.

Top ten teams were:

1. Bulgaria (165 out of 240)
2. Romania (151)
3. Turkey (131)
4. Serbia (122)
5. Italy (120)
6. Kazakhstan (105)
7. Moldova (97)
8. UK and Ireland (73)
9. Greece (59)
10. France, Macedonia A (38)

The top individual scorer was a student from Romania, who got 39 out of a total of 40 marks. Congratulations to Bulgaria and Romania.

Tradition has it that the UK leader to international olympiads repays the honour of that role by sharing experiences in a short report. So here are some recollections of BalkMO 2008 for those who were not lucky enough to be there.

To Ohrid (3-4 May)

The location of the 25th Balkan Olympiad was the beautiful and historical town of Ohrid, in the south western corner of Macedonia. Ohrid was famous in medieval times for having 365 churches, one for every day of the year. Much of the old town is well preserved, and its appeal is further enhanced by a stunning situation on Lake Ohrid, a crystal-clear body of water measuring twenty miles by ten and surrounded by low hills on all sides. Ohrid and Lake Ohrid are a UNESCO World Heritage Site, and anyone who has been there will endorse that designation.

Unfortunately, travelling there from the UK is not quite as easy as one might hope. Flights to Ohrid's own airport are rare; and there are no direct flights at all from the UK to Macedonia. A few weeks before departure, Jacqui and I ponder travel options with UKMT colleagues. Those of us who favour the land route from Tirana are shouted down, and we opt to fly to Skopje via Budapest, and on from

Skopje to Ohrid by coach. It means a crack of dawn start from Heathrow on 4 May, but it's the best option of an indifferent bunch.

In order to make the early start a bit more bearable, we will all stay at the Heathrow Novotel the night before departure (apart from Galin, who will be joining us at Budapest). I was deputy leader of the UK IMO team in 2005, but I've been out of the loop of UKMT activities since then, so it's only when we convene in the hotel foyer that evening that I get my first chance to meet Jacqui and the team. The six were picked just three weeks earlier, and a couple of correspondence sheets with them are all I have been able to fit in by way of preparation. Their work on these has been extremely promising: many of them would certainly grace a UK (or Irish) IMO team. Now that I meet them I am pleased to see that they are also a hardy crew for whom hitch-hiking from Tirana to Ohrid would have been as nothing. An opportunity missed there.

The following morning we fly out of Heathrow's new(ish) Terminal 5, which seems to have resolved all teething problems. Jacqui and I are relieved to rendezvous successfully with Galin at Budapest (phew!), and the whole team travels the final leg to Skopje together in good spirits, in spite of the problem sheets that I have inflicted on them for the journey. The Macedonian national water polo team are also on the flight.

Arriving in Skopje Alexander the Great airport in mid-afternoon, we are met by the efficient local organisers to tell us where to go (phew again!). We wait a lazy hour or two in the sunshine of a cafe outside the airport, while other teams arrive. Jacqui tries to learn the Macedonian for "I am a vegetarian". A spirit of scientific enquiry leads me to sample the Macedonian red wine, which proves to be not bad at all.

The Kazakhs are the last to touch down; and when they emerge from the airport it's time to head for Ohrid. Although the distance between Skopje and Ohrid is only 170km, our coach somehow manages to take 4½ hours to do the journey. On the plus side, we get the chance to enjoy the attractive countryside, which is wooded and hilly. But it's long after sunset before we reach the Hotel Metropol in Ohrid where the students and Jacqui will be staying. The leaders are accommodated on a separate site to help keep the exam paper secure. So after a final dinner with the team I am driven off to my hotel a few minutes further on. On arriving there after 11pm, I am given the shortlist of problems from which the exam will be chosen. The first jury meeting to start setting the paper will be at 9am the following day. I'd love to be able to report that I sat up for hours and solved all the shortlisted problems. But it's been a long time and lot of miles since we left the Heathrow Novotel before dawn. I hit the sack.

The problems (5 May)

The following day the jury of team leaders has a daunting task: to choose the problems for the exam paper from the shortlist of problems; after that to settle the exact wording, in English, of the problems; and finally to agree translations of the paper into all the different languages spoken by the contestants. While the BalkMO jury conducts its business in English (fortunately for me), the contestants write their solutions in their own native languages. As the leader of a guest nation, I don't speak or vote on the jury, but if I think I have a point that really needs to be made I can always tap the elbow of the Turkish leader next to me and bring it to his attention.

The Balkan Olympiad consists of one paper with four problems. The setters of Olympiad papers normally try to include problems from all the different areas of Olympiad maths (which is usually broken into four broad topics - algebra, combinatorics, number theory and geometry). The BalkMO jury normally try too to set problems of increasing difficulty: one “easy” problem as number 1, two “medium” problems as 2 and 3, and one “hard” problem as number 4 (where these terms should be understood to be relative: by any normal criteria, the problems that appear on the paper are all incredibly hard). All this means that when it comes to setting the paper, the jury is working within tight constraints, since only a small proportion of ordered quadruples of problems from the shortlist satisfy the desired criteria of variety and difficulty. By the time the jury convenes at 9am, I’ve had a bit of a look at the shortlist. As usual, the problems have been grouped into the four categories Number Theory (numbered NT1 to NT6), Algebra (A1 to A7), Combinatorics (C1 to C4) and Geometry (G1 to G8). Unlike IMO shortlists, the problems have not been put in (the Problem Selection Committee’s estimate of) order of difficulty within categories, so we are left to judge for ourselves how tough they are. This is not always an easy task. A problem can have a short and simple solution that leads the setters to think the contestants won’t find it too tough. But a short and simple solution can still be very hard to discover, and over the years olympiad papers have been littered with problems that the setters thought would be easy, but which turned out in practice to be far from it. Have a look at the first problem from this year’s BMO 2 paper for an example; or IMO 1996/5.

From what I can see we’ve got plenty of good geometry problems, but not quite so many strong possibilities in other areas. And we've got a relative dearth of good "hard" problems to choose from. A lot of the problems, naturally enough, have a distinctly Eastern European mathematical character – dependent on quite a heavy weight of symbolic notation, and often linked to problems from real analysis that we don't study until university in the UK. I'm not quite sure how well our students will get on with problems like that, but if any make it on to the paper it will be a good test for them (in the end, problem 2 of the paper was an example of that type, and some of our team rose to the challenge well).

Vesna Manova-Erakovic of the Macedonian Mathematical Society is our excellent chairman of the jury, and she keeps proceedings going at a good clip. First up we pick a hard problem to be Problem 4 on the paper. The outstanding candidate is a number theory problem NT2 about a sequence of integers, and it goes on with only minimal opposition. Next we go for the 'easy' Problem 1. There’s much more choice here. I have a soft spot for A7 (a smart inequality question), but attention soon focuses on an interesting triangle geometry question G2. It's a very nice result and a lot of approaches work. I wonder whether it's not closer to 'medium' difficulty than 'easy'. Time will tell.

At lunch time the leaders zoom across to the students hotel for a brief opening ceremony for the competition, before hurrying back to reconvene the jury to select two 'medium' difficulty questions to be Problems 2 and 3 on the paper. Since we’ve already got Number Theory and Geometry, we’d like to pick one Algebra problem and one Combinatorics problem to fill these remaining slots. A couple of inequalities questions, A2 and A3, get support in the Algebra category. The jury settles on A2, which has a very analytic character. It’s one of those ones that I know won’t really favour UK and

Ireland, but it's a sound question. And for Problem 3 there's a good combinatorics question C3 with a bit of a number theoretic flavour. It looks just the job.

It has already been an intensive day, but work is by no means over. By the time all four problems have been picked it's about 6pm; now it's time for me to step up to the plate to help draft the official English wording of the problems. In this task I am joined by two colleagues - Dan Schwarz, the observer from Romania and Milos Stojakovic, the leader from Serbia - both of whom speak English at least as well as me. The French leader Claude Deschamps sits in to keep us on the straight and narrow. After an hour we are ready to present our efforts to the jury.

It becomes clear that Problem 3 is going to cause the most grief. Nikolai the Bulgarian leader is concerned that our version doesn't lend itself to translation into Bulgarian. So it's back to the drawing board. After 15 minutes of head-scratching and some philosophical discussions about what is meant by a vertex of a square, we arrive at an English version which looks to my eyes rather less clear than our original, but it will do.

Here's the paper we had settled on:

Problems of the 25th Balkan Mathematical Olympiad in Ohrid, Macedonia
(time allowed: 4.5 hours)

Problem 1.

An acute-angled scalene triangle ABC is given, with $AC > BC$. Let O be its circumcentre, H its orthocentre, and F the foot of the altitude from C. Let P be the point (other than A) on the line AB such that $AF = PF$, and let M be the midpoint of AC. We denote the intersection of PH and BC by X, the intersection of OM and FX by Y, and the intersection of OF and AC by Z. Prove that the points F, M, Y and Z are concyclic.

Problem 2.

Does there exist a sequence $a_1, a_2, \ldots, a_n, \ldots$ of positive real numbers satisfying both of the following conditions:

i) $\sum_{i=1}^{n} a_i \leq n^2$, for every positive integer n;

ii) $\sum_{i=1}^{n} \frac{1}{a_i} \leq 2008$, for every positive integer n?

Problem 3.

Let n be a positive integer. The rectangle $ABCD$ with side lengths $AB = 90n + 1$ and $BC = 90n + 5$ is partitioned into unit squares with sides parallel to the sides of $ABCD$. Let S be the set of all points which are vertices of the unit squares. Prove that the number of lines which pass through at least two points from S is divisible by 4.

Problem 4.

Let c be a positive integer. The sequence $a_1, a_2, \dots, a_n, \dots$ is defined by $a_1 = c$ and $a_{n+1} = a_n^2 + a_n + c^3$, for every positive integer n. Find all values of c for which there exist some integers $k \geq 1$ and $m \geq 2$, such that $a_k^2 + c^3$ is the m^{th} power of some positive integer.

Sitting the paper (6 May)

The contestants sit the paper the following morning. This means that the quarantine between the leaders and students is no longer necessary, so we check out of our hotel after breakfast to move into the students' hotel. On arriving I'm pleased to meet up with Jacqui again. We both cross our fingers that our team are getting on ok. There's nothing we can do now but wait, and it's a long morning for us.

When the team finally re-emerge from the exam hall, the news seems to be reasonably positive. Most of them think they have solved at least one problem solidly and made some headway on at least one more. But Rong has had a bad day. He has spent a long time working on the tricky geometry problem, but hasn't cracked it. Anyone with experience of mathematical olympiads will tell you that on any one day success depends as much on luck as it does on ability and skill. Rong is not the first excellent British student for whom things haven't turned out as well as they might at the IMO or Balkan Olympiad.

Even so, this sounds like a pretty good performance overall, and I am cautiously pleased. I won't really know though until I get to see the exam scripts. They have been whisked away by the Macedonian organisers to be photocopied. There's nothing we can do till we get them back so the team, Jacqui and I head into Ohrid town for a stroll and dinner. The team have me and Jacqui perplexed with a guessing game in which they communicate to each other the identity of a celebrity of our naming, by use of only a few stocks phrases and clicks of the fingers. Peter, Rong and Hannah seem to be the experts, and Craig manages to work out the trick too. But to this day I have no idea how it is done.

Getting back to the hotel after dinner I find the scripts are ready, and settle down for a long evening poring over them in the hotel reception. The geometry is easy enough to mark. Craig has bashed it out by co-ordinates. Galin has an exquisite solution using the Butterfly Theorem[1]. Hannah has a slightly eccentric 3-page trig bash. For a start she has mis-read the question and got a different diagram. Secondly she appeals to something that she calls a special case of the Cosine Rule in which one of the angles of the triangle is 90 degrees. Closer inspection shows that this is Pythagoras's

[1] Given a chord *PQ* of a circle with midpoint *M*, let *AB* and *CD* be any two other chords passing through *M*. Let *AD* and *BC* meet *PQ* at *X* and *Y* respectively. Then *M* is the midpoint of *XY*.

Theorem. Nonetheless her solution is solid as a rock and she should get full marks. As I suspected, our team haven't found this problem especially easy.

The only substantive attempt on problem 3 is Andrew's. He has a nice method, which has run into the ground a little bit at the end.

Problems 2 and 4 are a bit trickier to judge. All the team have tried to approach the inequality with a variational method. This is a reasonable thing to do, but they have struggled to write it up convincingly. Andrew has pretty much nailed it. Peter and Hannah should certainly get something for their efforts. Both of them have produced rather meta-mathematical contributions, however, containing a lot of words and not many symbols. They are more like essays describing solutions of the problem than actual solutions. If it was hard for me to get my head around them, imagine how hard it must be for the Macedonian marking team, who don't have anywhere near as long, and whose first language is not English.

At 2am, I give in and crawl off to bed, leaving the reception deserted.

Coordination (7-8 May)

The following day the students head off on an outing to Ohrid, but it's time for Jacqui and me to earn our tickets by trying to get our team all the marks they deserve for their exam scripts. At the BalkMO (and IMO), marking is done by a process called coordination, in which each team's leaders sit down with experts from the host nation to discuss their students' scripts and agree on a mark for them. At IMO problems are marked out of 7. At BalkMO, it's out of 10, but the principle is exactly the same. The quality of coordination is one of the make-or-break criteria by which an Olympiad is judged. Ideally, it shouldn't be an adversarial process, though legitimate differences of opinion between leaders and coordinators about what a script is worth will always be common. The guiding approach is normally that marks should be awarded for progress towards a complete solution. But what counts as non-trivial progress, and what proportion of a full solution a student has been able to complete are more often matters of judgement rather than fact.

Happily Jacqui and I find that the quality of coordination at this Olympiad is excellent. Many of the coordinators are recent alumni of the Macedonian IMO team. And many have been on the Problem Selection Committee for this Olympiad, so they know the problems inside out. On the night after the exam, they have all stayed up until 6am working on scripts. That's the kind of effort that makes a big difference to the Olympiad.

First up for us for coordination is Problem 3: this is our weakest question and it's only Andrew who has got the backbone of a solution. After we explain how to fix one or two faults in his argument, we agree on 6/10, which seems about right. Galin, Craig and Peter pick up a few scraps.

So far, so good; but things get a bit trickier when we move on to the dreaded Problem 2 after dinner. Andrew will get 10 minus epsilon; but Peter's and Hannah's scripts have "long hard coordination" written all over them.

We troop in to the coordination room at the appointed time, but the Macedonian coordinators know what is coming and decide that they need to have dinner before they can face these British scripts. So an hour later we reconvene to try and unpick the UNK and IRL efforts.

Andrew picks up a very fair 9/10 for his nice solution. Then I set out the merits of Peter’s work (“He has all the ideas of a beautiful solution”). The coordinators look a bit baffled. They enumerate some of the deficiencies of Peter’s work (“This is not mathematics”). Maybe I look baffled (“You have to understand that British students don’t study these topics until university, so he doesn’t always express himself in the conventional way”). And so it goes on. Eventually we shake hands on 4 marks, and no one could argue with that. After another debate we agree on 4 marks for Hannah as well, by managing to persuade the coordinators that she had hit upon most of the key ideas for an interesting new solution (which was true!). We’re very grateful for the care the coordinators take over these difficult scripts, particularly given that it is now getting deep into the evening. Rong picks up a welcome mark for having the first key idea for the variational solution.

For some light relief, we knock off coordination of Problem 1 before going to bed. This is easy, because half the team have done it, and half haven’t really gotten anywhere. The coordinators make positive noises about the beauty of Galin’s solution using the Butterfly Theorem; less so about Hannah’s ink-intensive trigonometric argument.

We polish off coordination of problem 4 early the following morning. The full results of the UK and Ireland team are as follows:

UNK & IRL 1	Galin Ganchev	10	0	2	7	19
UNK & IRL 2	Andrew Hyer	1	9	6	0	16
UNK & IRL 3	Peter Leach	0	4	2	6	12
UNK & IRL 4	Craig Newbold	10	0	1	0	11
UNK & IRL 5	Hannah Roberts	10	4	0	0	14
UNK & IRL 6	Rong Zhou	0	1	0	0	1

That means we have a total of 73 out of 240 marks: down on last year, but on a significantly harder paper.

Common practice at mathematical olympiads is for the marks to be displayed on notice boards as they are agreed during coordination. Everyone tries hard to guesstimate individual and team rankings, and medal boundaries, from the partial data as it emerges. UK and Ireland’s results are complete by mid-morning, but most other leaders still haven’t agreed all their marks with the coordinators; so we still can’t be sure what medals we will get. Like the IMO, the BalkMO awards medals roughly in the ratio 1:2:3 for gold: silver: bronze. But while the guide at IMO is that no more than ½ of competitors should receive medals, at BalkMO a more generous 2/3 can expect to get them.[2]

Even on the basis of the results on the boards already, it’s obvious that the paper has been found to be pretty tough across the board. By the time the team return from their outing at lunch time it’s

[2] Medal boundaries are determined at BalkMO by consideration only of the scores of full participant teams; medals for competitors from guest teams are then awarded based on the same medal boundaries.

clear the bronze boundary will be in single figures. So it will be five medals for UNK and IRL this year. Even better: as more data comes in, it looks like Galin will get a silver. This is confirmed by the final jury meeting that evening which signs off all the marks from coordination and sets the medal boundaries for bronze, silver and gold at 5, 17 and 29 respectively.

After the results (9-10 May)

We have one more full day in Ohrid, culminating in the medal ceremony and closing banquet. That morning the party splits. Galin must sit the Irish National Olympiad Papers (2 x 3 hours) in order to win a place on their IMO team. We find a quiet room in the hotel Metropol for him to tackle them. In a quite astonishing display of camaraderie, Peter agrees to sit the papers along with Galin. The rest of the team are not made of such stern stuff and spend the day on the boat trip on the lake. I go into town to buy provisions to get Peter and Galin through the ordeal. It has been said that a mathematician is a machine for turning coffee into theorems. But in Peter's case the raw material seems to be chocolate. I return with 8 bars in the hope that this will see him through 6 hours of exams.

The following day is the time for partings. We say goodbye to our new Macedonian friends after breakfast; and au revoir to Claude Deschamps and his French team, with whom UNK and IRL have struck up excellent relations, at Skopje airport. Galin peels off at Budapest to return to Dublin; Andrew, Hannah and Rong go their own way at Heathrow. Peter, Craig, Jacqui and I have one more night at the Novotel before wending our separate ways too, the end of one more olympiad adventure.

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Thanks...

Many thanks to the UK and Ireland team for all their efforts, and for representing themselves and their countries with distinction. I'm sure that they would wish to join me in thanking Jacqui Lewis for her wonderful company and constant good sense throughout.

Thanks to the nations of the Balkan Mathematical Olympiad for their kindness in extending an invitation to the UK to participate in their splendid competition.

And last but by no means least, thanks to the Macedonian Mathematical Society for organising such a hospitable and professional competition, and for sharing with us some of the treasures of their beautiful and historic country.