REPORT ON THE 35TH INTERNATIONAL MATHEMATICAL OLYMPIAD HONG KONG : 8 - 20 JULY 1994 AND ON RELATED NATIONAL EVENTS WITHIN THE UK

We hope this report will be of interest to all who enjoyed this year's

- **British Mathematical Olympiad** *(for those in their last years at school),*
- **International Intermediate Invitational Mathematical Challenge** *(for School Years 10-11 - English style),*
- **UK Junior Mathematical Olympiad** *(for School Years 7-8).*

We ask teachers to make copies available to all those who might be interested.

General Background

What is the IMO? Each year, in July, students from many different countries meet to take part in the *International Mathematical Olympiad* - or *IMO* as we shall call it. Participating countries can send up to six students. The problems are tough, so most students are in their last year or two at school. This year's event was held in Hong Kong and attracted 385 students from 69 countries - including 6 from the UK.
Candidates have to sit two 4 1/2 hour papers on successive days, with three problems on each paper. Usually only a handful of students manage to solve all six problems, so the very best students might hope to solve each problem in 1 1/2 - 2 hours. (This year's problems are given on page 5: pick a problem that looks interesting and have a really good go at it. When you have got as far as you can, put it aside for a while and come back to it later. Finally send your best attempt *enclosing a stamped, self-addressed envelope* to Tony Gardiner, School of Mathematics, University of Birmingham, Birmingham B15 2TT)

Why is the IMO held? The *IMO* has three *official* goals: (a) to encourage able young mathematicians, (b) to give such students an opportunity to meet, and (c) to allow the Leaders to discuss and compare national education systems. The first goal should not be misunderstood: the *IMO* does not pretend to reach *all* able young mathematicians - though it does manage to stimulate many more than those few who are selected to represent their country. Mathematicians develop in many different ways. Some blossom early and need to be stretched. For most talented mathematicians the crucial years are those between the ages of 18 and 25; they may not stand at school in quite the same way, but they can still benefit from the experience of grappling with hard *Olympiad* problems. For those who do stand out and who attend the *IMO* one can only ask that the experience should contribute positively to their development: the *IMO* is meant to encourage and stimulate; it is not a guarantee of future success.

Nevertheless it is encouraging to see that the *IMO* can indeed encourage and stimulate future mature mathematicians. Mathematics conferences are mostly specialist affairs. But every four years there is a huge jamboree, the *International Congress of Mathematicians*, where 3000 or more mathematicians gather to hear about the most interesting developments of the last four years in all areas of Mathematics. I have just returned from the latest such meeting in Zürich, where 16 people had been invited to give hour long plenary lectures and 160 or so had been invited to give 45 minute specialist lectures. Six of those invited to give specialist lectures were not only from the UK, but were also young enough to have possibly taken part in Olympiads when they were at school. To me it was very striking to discover that four of the six

had represented the UK at an *IMO*: AJ Wasserman (1974), Richard Borcherds (1977), Richard Taylor (1980), and Tim Gowers (1981). On the other hand, two who received the greater honour of being invited to give plenary lectures - Paul Seymour (who has just reproved the *Four Colour Theorem* and much more) and Andrew Wiles (who has not yet managed to complete his 'proof' of *Fermat's Last Theorem*) - were 'missed' by school-based Olympiads! (Similarly, three of this year's UK *IMO* team were Gold Medallists in the UK JMO in 1991, while two never took part and one claims to have done miserably badly.)

Nevertheless the real justification for the *IMO* is *unofficial* and has nothing to do with the event itself: the *IMO* provides an incentive for each country to establish a pyramid of activities for *masses* of interested students, all of whom can benefit even though they will never take part in the *IMO*! Thus the *IMO* should be seen as just the tip of a very large, more interesting, iceberg.

How does the IMO help the mass of bright students? National Mathematics 'competitions' developed originally in Hungary, where they have been going strong for over a 100 years. Hungary now has a range of events, and everyone who is interested in Mathematics can get involved in some way. Most important of all, they started a *problem-solving journal for students* in 1894. This journal - affectionately known by its abbreviation *KöMal* - is still going! The important thing to note is that these events, and KöMal, are not just for a few brilliant students (if they were, they could not have survived); instead they challenge and encourage large numbers of ordinarily bright youngsters.

One of the most obvious effects of this Hungarian 'pyramid' is that since 1900 Hungary, with a population of around 10 million, has produced an astonishing number of brilliant mathematicians, and an equally striking number of very good mathematicians of the second rank. (A successful system has to produce more than just Nobel prizewinners!) The Hungarian example was copied by the Soviet Union in the 1930s and by the whole of Eastern Europe after the Second World War. Each country then developed a system with its own local character.

The *IMO* started in 1959 as an event for countries in the Soviet bloc and helped to stimulate the development of national events, journals, local weekly mathematics circles, summer camps, etc. involving large numbers of students. The achievements of many of the smaller countries were truly remarkable, but Hungary stands out: each year from 1959 (when there were 7 countries) to 1975 (when there were 17 - including the UK) they finished first, second or third! In 1965 Finland was invited as the first non-communist participant, and in 1967 the UK, France, Sweden and Italy were invited. Since then numbers have grown steadily (with 23 countries in 1979 when the *IMO* was held in London, 38 countries in Finland in 1985, and 69 countries this year).

In 1993/4, for the first time, secondary schools in the UK had available a range of national events designed to challenge interested youngsters of all ages. On each level there is now a large (and rapidly growing) multiple choice paper for a large group of students:

the UK JMC (Years 7-8);	the UK IMC (Years 9-11);	the NMC (Years 12-13).
(98 000 entries)	(105 000 entries)	(30 000 entries)

These events aim to provide an unusual, and enjoyable, challenge for large numbers of students, and to provide modest rewards which may help encourage them to aim that little bit higher.
As part of this general strategy of seeking to encourage able youngsters, 500-1000 participants on each level are invited to take a subsequent written paper

the UK JMO (Years 7-8);	the IIIMC (Years 9-11);	the BMO (Years 12-13).

These written papers are quite different from the multiple choice papers, both in style and in purpose. Most of those who take part at this level are used to getting all their schoolwork right without too much difficulty. This can mislead students into thinking that Mathematics is easy. It is not! The problems on these written papers are meant to be accessible, but they require

candidates to think, to piece together a complete solution to a small number of problems (the first time you take such a paper you should aim to complete just one or two problems), and then to present their solutions clearly. In Mathematics it helps to be clever, but it is more important to develop tenacity - the determination to work away at a problem until you see how to solve it completely. As in the fable of 'The Tortoise and the Hare', the student who is not satisfied with just giving an 'answer' which seems to work, who never gives up, and who is determined to find a complete solution can often outrun those who think they are 'quicker'.

In addition to the above events, we now have two residential meetings:
- the three day *Olympiad Training Weekend* for those in Years 12-13, held in Trinity College, Cambridge in April, and
- the *Mathematics Summer School* for those in Years 10-11 (plus a few from years 12-13), held in The Queen's College, Oxford in early July.

In the long run, success in Mathematics depends on students being willing to work on their own. This is one reason why we try to arrange for those who do well at the highest levels to have the chance of some correspondence work. However, what is really needed is to somehow ensure that solving hard problems becomes a natural, and accepted, part of adolescent mathematical life. It is good to see that we now have a number of journals with problem corners: e.g. *Plus* for younger students, *Mathematical Spectrum* and the Student Problem Corner of the *Mathematical Gazette* for older students. But we do not yet have anything to match the journals available in Eastern Europe. Any teachers who would be interested in helping to create such a journal should contact Tony Gardiner, School of Mathematics, University of Birmingham, Birmingham B15 2TT.

Preparing for the 1994 IMO

How is the UK IMO team chosen? As a result of the *British Mathematical Olympiad Round 1* in January (for 650 or so), 100 students were invited to take *Round 2* in February. We then chose 20 students (including 10 or so younger students) to attend the three day *Olympiad Training Weekend* in Trinity College, Cambridge in April. The weekend includes a 4 1/2 hour Olympiad-type paper, after which the UK *IMO* team of six-plus-reserve is chosen. This year's Magnificent Seven were:

Ed Crane	(Colchester RGS;	contact teacher Dr G Davey)
Andrew Cruise	(Sherborne S;	contact teacher R Ambrose)
Matthew Fayers	(Wilson's S;	contact teacher D K Pearson)
Ben Green	(Fairfield GS;	contact teacher F Burke)
Catriona Maclean	(Harrogate GS;	contact teacher B W Crosswaite)
Joseph Myers	(Rutlish S;	contact teacher L I Makinson)
Jacob Shapiro	(Westminster S;	contact teacher M Davies)

The Leader this year was Tony Gardiner (University of Birmingham), and the Deputy Leader was Vin de Silva (Trinity College, Cambridge) - who was a member of the UK *IMO* team in 1989 and 1990.

How does the team prepare for the IMO? Most of the countries that land up in the top 15 or so at the IMO have residential 'training' camps - for one or two months (or more). There are those who feel that this level of preparation is excessive. Fortunately in the UK we have no choice since public examinations dominate the period from early May right up to the very end of June. Nevertheless it is important that those who go to the *IMO* should do themselves justice. Thus, in addition to the work students will have done in the process of being selected, some fairly serious preparation has to be done between April and July. Those selected somehow have to fit in a tough schedule of *IMO* correspondence problems, in addition to their ordinary school and exam work.

The team then meets for a few days (this year at the *Mathematics Summer School* in Oxford) just before leaving for the *IMO*. This may be the first time

the students have met *as a team*, but they have already been suffering together (ringing each other up when necessary) for two months! Success at the *IMO* depends on how the team knits together, not just on how individuals perform.

Who else is involved? Hundreds of people! As indicated above, no school could take part if some teacher in that school was not willing to do the work of entering students and helping them to prepare. The papers would never be set if there were not teachers and mathematicians willing to give up their time to devise and to select the questions. The results would never appear if bunches of individuals did not abandon family and friends to mark the scripts and produce results lists. Prizes sometimes arrive late, but they would *never* arrive at all if the necessary work was not done. Residential meetings depend on tutors, and on places - like Trinity College (Cambridge) and The Queen's College (Oxford) - where we can meet. And though all the work is voluntary, everything costs money, so we could not continue without sponsors.

But the most important part of any event are the participants. So we would like to think that the thousands of students who have taken part at various levels of the 'pyramid' this year feel that they too have 'been involved'.

The 35th IMO

How did we get on? The answer will depend on what one expects! My own priority is that students who represent the UK should find the experience not only challenging (i.e. hard), but also rewarding: that is, they should feel afterwards that they have achieved something. In an Olympiad problem-solving competition that means that each student should aim to solve at least one problem completely (out of three) on each day. That is easy to say, but much more difficult to achieve. Faced with problems of a kind one has never seen before, with the clock ticking away, it is only too easy to get stuck, or to spend a long time on an approach which turns out not to work.

Each problem is marked out of 7 - and there are no sympathy marks. A perfect solution will score 7 marks, even if it includes the odd slip of the pen. A perfect solution which omits some key point will probably score 6 or 5 marks. An intelligent approach *which does not work* may well score only 0 or 1. The judges are looking for *solutions*; they are not there to reward bright ideas which *don't work*. The one exception is that a bright idea *which could have worked* does get rewarded: in such a case, a partial solution which was simply not completed may score 3 or 4 marks, depending on how much is missing.

In the event the UK team did (almost) everything I asked of them. One thing I found particularly pleasing was the variety of successful approaches they found for each question: in the absence of an intensive 'training program' students cannot just apply taught methods but must use their own ingenuity. Here are their scores, question by question.

	Q1	Q2	Q3	Q4	Q5	Q6	Total	Medal
Ed Crane	4	6	7	2	3	7	29	Bronze
Matthew Fayers	4	7	2	3	6	2	25	Bronze
Ben Green	7	7	7	7	7	0	35	Silver
Catriona Maclean	7	7	7	7	7	7	42	Gold
Joseph Myers	7	6	7	7	6	7	40	Gold
Jacob Shapiro	7	7	7	2	6	7	36	Silver

The cut-off scores for medals were set as follows:
Gold ≥40 (30 students); Silver ≥30 (64 students); Bronze ≥19 (98 students)
The team total of 206 put us in 7th place, one point behind Vietnam. The top twenty were as follows: USA (252), China (229), Russia (224), Bulgaria (223), Hungary (221), Vietnam (207), UK (206), Iran (203), Romania (198), Japan (180), Germany (175), Australia (173), Poland, S Korea, Taiwan (all 170), India (168) Ukraine (163), Hong Kong (162), France (161), Argentina (159).

Here are this year's questions: remember 4 1/2 hours are allowed each day.

FIRST DAY

1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j, $1 \leq i, j \leq m$, there exists k, $1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \ldots + a_m}{m} \geq \frac{n+1}{2}.$$

(France)

2. ABC is an isosceles triangle with AB = AC. Suppose that

(i) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if QE = QF.

(Armenia+Australia)

3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

(a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

(Romania)

SECOND DAY

4. Determine all ordered pairs (m,n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

(Australia)

5. Let S be the set of real numbers strictly greater than -1. Find all functions $f: S \longrightarrow S$ satisfying the two conditions:

(i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;

(ii) $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$

(UK)

6. Show that there exists a set A of positive integers with the following property: for any infinite set S of primes there exists some $k \geq 2$ and two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S.

(Finland)

Remember to write up and send in your solutions - enclosing an SAE to: Tony Gardiner, School of Mathematics, University of Birmingham, Birmingham B15 2TT.

How is the IMO organised? A country offers to host the *IMO* several years in advance. The host country thereby commits itself to inviting all countries who took part in previous years (and any new countries it chooses). They also accept responsibility for all costs "from touchdown to take off". Thus other participating countries only need to cover the cost of preparation and travel. This system works remarkably well. Surprisingly, the list of willing hosts stretches to 2003! The next three *IMO*s are in Canada, India, and Argentina.

Most organisational details have to be planned well in advance by the host country. However, the *academic organisation* is the responsibility of the *IMO Jury*. Each team has a *Leader* and a *Deputy Leader*. The *Leaders* form the *IMO Jury*, which meets three days before the teams arrive - to select the problems, to agree on the exact wording, and to approve the translations (into nearly 50 languages!). Each *Leader* has one vote. Given the many languages and the conflicting interests of different countries it is a miracle that the system works at all. The system has its flaws; but it works - up to a point.

The *Deputy Leaders* travel with their teams. Great care is taken to keep the *Leaders* (who know which problems have been chosen) separate from the teams.

How are the problems chosen? Each participating country is asked to submit up to five problems. These form a pool of 150+ questions. A *Problems Committee* selects a shortlist of around 30 of the best problems submitted, and the *Jury* then uses this as the basis from which the six *IMO* problems are selected. If they wish, they may ask for additional problems not on the shortlist.

In each of the last three years one of the questions submitted by the UK has survived this selection process. We will be lucky to maintain this record next year. The origin of each of this year's problems is given on page 5.

In the end, the problems selected were somewhat easier than we would ideally have wished. Many people imagine that countries who do very little 'training' should vote for easier problems. That is a mistake. Hard problems allow bright but untrained students to compete with those who have done lots of training; problems of medium difficulty can seem impossible to many students and yet be solved in a routine way by those who have been well trained.

There were all sorts of difficulties with the problem selection this year. It is traditional to try to make Question 3 and Question 6 genuinely tough, but the Jury did not feel it had enough good hard questions to choose from. Question 3 proved far too easy (with 8 countries achieving the maximum score possible of 42). This left only one intentionally hard problem - Question 6 (on which the USA scored 42, Hungary scored 34, Russia scored 31, the UK and Iran both scored 30, Taiwan scored 29, Bulgaria scored 28, Romania scored 27).

On the other hand the first question on each day is meant to be more accessible, but the Jury's judgement in such things is not always reliable. Thus Question 1 proved to be far from 'easy' and the UK team score on this problem of 36 was surpassed only by the USA and China (both with 42).

Snapshots of the 35th IMO

Thursday 14 July: It is 8 o'clock in the evening after the Second Day's paper. Vin and I have just entered a crowded room on the 30th floor of the *Kowloon Panda Hotel*. Through the window we can see the evening lights of *Tsuen Wan*. We have come to wait our turn to present our students' solutions of *Problem 3* from the First Day's paper to the judges.

Judging started at 4 pm and is already way behind schedule. We are in for a long wait in a very noisy room, though we do not yet know this. We urgently need some peace and quiet to work through the scripts which we will have to present to the judges tomorrow morning, but we dare not escape to our rooms in case we miss our turn.

I am feeling both excited and nervous. I am excited because, having worked

through all the scripts from the First Day, it looks as though the team is doing exceptionally well - dropping only 13 marks or so: (I was right to be excited: only the USA, China, Bulgaria and Hungary did better on the First Day, though there was no way we could have known this). I am nervous partly because I have no way of knowing how the other teams have done, and partly because I do not yet know how the team fared that very morning on the Second Day's paper. Most years we would meet and talk to the team after the end of the Second Day's paper, but this year the organisation kept us apart.

On the way up to the 30th floor, I visited the *IMO* Office to collect the Second Day's scripts. I am now clutching them nervously. I would love to peep, but try to concentrate on the immediate task (scheduled for 8 pm!) of presenting the team's solutions to *Problem 3* to the judges.

We take our seats and quietly rehearse the order in which we plan to present the scripts, the possible sticking points which we must be prepared to explain, and any other points we intend to stress. We soon discover how long we will be waiting: (we eventually finished after 10.30 pm). After a brief exchange of glances, I sheepishly open the package of Second Day scripts.

Our curiosity is natural, but it is not clear how we can hope to interpret what we find. All six students seem to have 'solved' at least one problem. And it looks as though Katy has achieved the perfect score we had quietly hoped for - though we will have to work hard on the details to make sure that the judges do not find some unexpected flaw. (We could not then know that 22 students would in fact achieve a perfect score - including all six members of the USA team.) As far as we can see Joseph too has had a pretty good Day 2 (though it later transpired that he had missed one minor case in *Problem 5*). We knew that *Problem 6* would be something of a hit-or-miss affair, and this is confirmed when we find that, in addtion to two beautiful solutions to *Problems 4* and *5*, Ben has returned an empty folder for *Problem 6* - contrary to our team target of scoring "*no zeros*". (We did not know then that Ben had in fact struggled for two hours on *Problem 6* without success. The *IMO* is tough!) Still, it looks as though Ben and Jacob will both score 35 or so, which looks good. (Most years such a score would be sufficient to earn them both a Gold medal. This year is different, but we could not have known that.)

Tuesday 19 July: It is 3.30 pm and we are back in *Sha Tin* Town Hall - a huge modern Concert Hall with seating for 1500 people. Down the side of my seat I find a program for a primary school prizegiving that very morning: the Hall is clearly well used by the local community. All around me there is a festival atmosphere. The organisers would like things to go like clockwork, but this is more like an oriental market place! One hundred and ninety two students go up to receive their medals (in batches of six or twelve) - all cheered on by their teams. One delightful Canadian, wrapped in a maple leaf flag and topped out by a straw hat, leaps up at the front each time a Canadian's name is called. Many medallists carry team mascots or national flags. And all pose for "that photo" to show the folks back home. If this were *Buck House* the Queen would have a fit. Yet, despite the chaos, it is also somehow dignified.

Vin and I are festooned with cameras from the team and positioned to "do the business" on their behalf. Our six are all in different batches, but each carries one of our two *Dwarf* mascots when they go on stage, before passing it down the line. (Inexperienced observers may think they - the *Dwarfs* (or is it *Dwarves?)*, not the team members! - look like garden gnomes. *We* know better.

Wrapped around this celebratory chaos is the formal part of the ceremony - with speeches, wonderful Chinese music on instruments I have never seen before, and dancing. The Opening Ceremony was presided over by the Governor, Christopher Patten, who gave a relaxed, and thoroughly impressive speech. In contrast, the Closing Ceremony has speeches from senior "representatives of the China News Agency" - code for Beijing's local powerbrokers. In this way the Hong Kong *IMO* reflects the delicate balancing act which is Hong Kong 1994.

After the ceremony we all adjourn to a local restaurant for the *Farewell Dinner*. (It is astonishing how many restaurants in Hong Kong can swallow up 6-700 people without flinching. One restaurant had so many large rooms that I completely lost the IMO party of 600 or so: it must have had room for 2 000 people.) As usual it takes an hour or more before any food appears: we are waiting for the guests! I had expected Hong Kong to be super-efficient, but punctuality seems to be almost unknown - at least by those in authority. Each team is at its own round table, and the delay provides a welcome opportunity to move round the room, chatting to other teams, congratulating individuals, making peace with those judges who had given us a hard time, and generally relaxing. After all the work of recent months, there is bound to be a sense of anticlimax. Ed has mixed feelings - having missed a Silver by 1 mark. And Ben is still rueing that empty folder for *Problem 6*. But the team has done its job and can be well satisfied.

I drift through the meal in something of a daze, though conversation is lively enough. The example set by the Canadian journal *Mathematical Mayhem*, which is edited and run by students, leads to the idea that we should try something similar. And when I mention that as yet no country has offered to host the *IMO* in the year 2002, the team promptly decide that the UK should put in a bid (and pledge their support). But when the meal ends, teams and *Leaders* go their separate ways in separate buses!

Tomorrow is departure day. All flights to Europe leave late in the evening so we have a whole day to kill. We plan a lazy day, with the team taking a bus to our hotel where we can leave the luggage until late afternoon, before finding our way to Kai Tak.

Thursday 21 July: It is 5.20 am and eight weary travellers enter Heathrow's Terminal 3 after a 14 hour, non-stop flight. The Terminal is amazingly quiet: it looks as though we are almost the first flight in that morning. The luggage appears fairly quickly. *Dwarf 1* is transferred from Ed's bag to mine. *Dwarf 2* is in Vin's case. We pass through customs and emerge into the arrivals area. There are no press photographers, no TV cameras! In fact there are very few people at all, other than Joseph's parents and sister Eva (Gold Medallist in Moscow in 1992) - all beaming shyly.

It is time for the team to take a well earned break. Mine will have to wait. I would like to hang around until the others have been collected, or have gone on their way, but I have just 24 hours to get back to Birmingham, arrange some newspaper coverage of the team's success, catch up with the family news, and sort out 3 weeks' worth of mail, before leaving for a conference in Bulgaria and then to the *ICM* in Zürich. So I make my apologies and set out for the Coach Station in the hope that there may be a 6 am coach to Birmingham. There isn't! And the first service at 7.15 fails to arrive. When trying to change my ticket I meet Katy, Jacob, Ben and Ed heading for the tube. It is an unsatisfactory way to say *Goodbye*. I eventually catch the 7.50 coach.

Back in Birmingham I manage to interest *The Daily Telegraph*, who produce an excellent report; their photo of 'Joseph' looks strangely like Jacob to those who know, but this is unlikely to worry anyone else!

This year's *IMO Celebration* will be held in the University of Birmingham on Thursday 29 September. Tea will be served in the foyer of the School of Education. The *Awards Ceremony*, and *IMO Lecture* will begin at 4.45 in the Vaughan Jeffreys Lecture Theatre and should finish by 6.15. The lecture will be given by Professor David Williams of the University of Bath with the title *'Randomness: The only source of order'*.
Schools who can get to Birmingham are welcome to bring groups of interested students and teachers (we suggest Years 11-13). It would help if they could phone 021 414 6580 a week or so beforehand to give an estimate of numbers.

Make sure your School/College enters the 1994 National Mathematics Competition