33RD INTERNATIONAL MATHEMATICAL OLYMPIAD

MOSCOW RUSSIA

10TH - 21ST JULY 1992

Each year the Leader of the UK IMO team writes a report for the UK organizing committee. Last year I made this report 'informal' in style, and circulated copies to all schools which had taken part in the BMO, hoping that those who had taken part would be interested in how the UK team had fared at the highest level. I also hoped that some (unknown) readers might be inspired to "Go for Gold" in 1992. To judge by the success of the 1992 UK IMO team, this strategy may have been more successful than I could have expected (read on!).

Last year's report was well received and we have again produced a report

- *which is informal in style;*
- *which summarises the whole sequence of events from the NMC up to the IMO, so that the reader can see how the different stages interact;*
- *which is circulated to all schools which took part in the BMO in the hope that they will make copies for interested colleagues and for students who might be interested in taking part in the coming year; and*
- *which is also circulated to all those who have contributed to the success of the 1991/2 UK Olympiad program: we are grateful to them all.*

It may help if we mention a few organisational details at the outset. (1) The National Mathematics Contest (NMC) is run by a committee of the Mathematical Association. Strictly speaking this has nothing to do with the UK Olympiad program. Since however, this is the way most students become aware of the British Mathematical Olympiad (BMO), it is important for us to retain strong links with the NMC. We are grateful to them for all their work. (2) The British Mathematical Olympiad (Rounds 1 and 2), the 3 day residential training session which takes place in early April, the selection and preparation of the UK team for the International Mathematical Olympiad, are all the responsibility of the British Mathematical Olympiad Committee (BMOC), which was set up as an independent group in 1991 by the Mathematical Association, the London Mathematical Society, The Edinburgh Mathematical Society, and the Institute of Mathematics and its Applications, with support from the Royal Society and the Department of Education and Science. The BMOC is grateful to all its sponsors - especially to Trinity College, Cambridge who have been a major supporter during the last few turbulent years.

Any readers who have ideas about, or would be interested in contributing to, the work of the BMOC, should write to me: Dr Tony Gardiner, BMOC Secretary, School of Mathematics & Statistics, University of Birmingham, B15 2TT.

General Background

For many countries the International Mathematical Olympiad (IMO) marks the climax of their own domestic program of Mathematical Olympiads. At the IMO each country enters up to six students. This year there were 323 students from 58 officially competing countries. There were also 28 students from 10 countries which were not competing officially (new countries which emerged from the break up of the old Soviet Union - Estonia, Lithuania, Latvia, Armenia, Azerbaijan, Byelorussia, Kazakhstan, Moldova, Turkmenistan, Ukraine).

It is easy to dismiss an event which involves only six students from each country as being irrelevant to most able young mathematicians. However, the students who represent each country are chosen from a much larger group - namely all those who take part in the various stages of that country's national Olympiad sequence. Indeed the success of many countries is roughly proportional to the number of students who take part at some level. In the top four countries at this year's IMO - China, the old Soviet Union, Romania,

and the USA - many millions take part in the first stages: the success of their IMO teams can then be enjoyed and appreciated by all who took part at some stage, whether or not they themselves made it to the very highest level.

Mathematicians come in all shapes and sizes, and develop in different ways and at different ages. But all of us can draw inspiration from the efforts and the successes of others. Olympiad problems - especially the experience of being completely stumped by harmless looking problems - should encourage all of us to aim a little higher. Those who succeed at one or other level demonstrate what they might achieve in the future; others catch a first glimpse of that new world of hard but worthwhile problems which may at present be beyond their powers, but which could be mastered, given time and effort.

The two best performances in this year's UK IMO team were both from students who during last year's selection process had found themselves slightly out of their depth. But both of them went away and worked on their own, and perhaps surprised even themselves by what they achieved: Mark Walters of The Weald School in West Sussex came an astonishing 11th overall in this year's IMO, while Eva Myers of Streatham Hill and Clapham High School delighted us all by coming 14th. Of such things are fairy tales made.

The other four members of this year's team also acquitted themselves superbly: Robin Michaels of Haberdashers Askes Boys School in Elstree and Luke Pebody of Rugby School came joint 47th, Oliver Johnson of King Edwards School in Birmingham came 87th, and Karen Page from South Bromsgrove High School came 94th. These are remarkable results, and show just what our students can do.

We hope that all those who struggled in the preliminary rounds of the 1991/2 UK Olympiad program will be able in some way to share in their success.

The UK Selection Process

On 22nd November 1991 25 000 students took the *National Mathematics Contest* - a 1 1/2 hour multiple choice paper for students in their last three years at school. 1 500 of these received Gold certificates, 3 000 received Silver certificates, and 4 500 received Bronze certificates.

On 15th January 1992 almost 650 of the best students took Round 1 of the *British Mathematical Olympiad* - a 3 1/2 hour paper with just five problems. Olympiad problems are not just hard A level problems: they force students to *think* in a truly mathematical way, and cannot usually be solved by merely applying the right standard method. For students who have been trained on a diet of A level papers, and who take it for granted that all mathematics problems are as predictable as A level questions, the experience of facing genuinely hard problems for the first time can come as something of a shock.

When faced with an unexpectedly hard problem, it is always tempting to give up too easily. But the important thing in any Olympiad (as in real Mathematics!) is *not* to give up, but to struggle on with the aim of trying to solve one or two problems *completely*. Those who managed this on the BMO Round 1 had already achieved something substantial. Thus, though the scores may have seemed to be rather low to those who were used to a different kind of mathematics, the extent of many students' achievement was recognised by the award of over 100 prizes to all students who scored ≥ 10 on the BMO Round 1.

In the light of each student's performance on the NMC and on BMO Round 1 (with most emphasis on the BMO score) we invited 60 students to take part in Round 2 - another 3 1/2 hour paper, this time with just 4 problems; this took place on February 13th 1992. The problems were more technical than those on Round 1, but they were still considerably easier than actual IMO problems (though they may not have felt like it to those taking part for the first time!).

Students invited to take Round 2 were not just the 'top 60', since some

allowance was made for age, for experience, and for the *quality* of the solutions presented on Round 1. Selection is always difficult. Limitations on numbers mean that we often have to choose between individuals who cannot easily be 'ranked'. We hope that those we leave out appreciate this, and find encouragement from what they *have* achieved alongside their disappointment at missing out. Several of this year's UK IMO team did rather badly on Round 1 of the BMO, and only began to show what they could do on Round 2! Others might well have done the same if only we had been able to identify them.

The next stage was to choose 20 students to take part in the 3 day residential "training" session held at Trinity College, Cambridge from April 2nd-5th. IMO problems are substantially harder than BMO problems, so some kind of further preparation is essential if students are to do themselves justice. However, there is a limit to how much one can achieve in just three days. And since the summer term tends to be dominated by public examinations, the UK team cannot imitate other countries by indulging in serious, intensive, residential training. We therefore have to find ways of encouraging potential team members to do their own preparation in their own time. This may explain why the word "training" in the previous paragraph appeared in quotation marks. The residential training session provides very valuable basic instruction in Algebra, Combinatorics, Geometry, Inequalities, and Number Theory, and general sessions on *How to attack problems which look horribly hard* and *How to write out solutions*. But if the session is to bear fruit, it must inspire students to go home ready to do a lot more work on their own.

On the last morning of the training session the students sit a 4 1/2 hour, 3 problem, IMO-type *Final Selection Test*, after which the IMO team of six (plus one reserve) is selected.

Many of those who are invited to the training session have one or more years left at school. We depend on these younger participants being sufficiently motivated to go home and do the necessary extra work for themselves. We are pleased to see that a number of these students who did not make the team have continued to send in solutions to problems throughout the summer. We hope that next year their sights will be set that much higher. But above all, we hope they will have benefitted mathematically from their work this year.

The 33rd International Mathematical Olympiad

This year's UK IMO team consisted of

Oliver Johnson (King Edwards School, Birmingham)
Robin Michaels (Haberdashers Askes Boys, Elstree)
Eva Myers (Streatham Hill & Clapham High School)
Karen Page (South Bromsgrove High School)
Luke Pebody (Rugby School)
Mark Walters (The Weald School, Billingshurst)

The reserve was Robin Bhattacharyya (Manchester Grammar School), the Team Leader was Tony Gardiner (University of Birmingham) and the Deputy Leader was Christopher Bradley (Clifton College, Bristol).

Having succeeded in our own Olympiad sequence, these students were faced with a much tougher challenge. The IMO problems they were to tackle on those two long mornings in Moscow in July would be harder than anything they had done up to April. Moreover, the IMO requires reliability as well as creativity. It is not enough to boast that one could *in principle* have solved this or that problem; neither is it enough to construct an almost perfect solution if one then makes a silly algebraic mistake in the last line (two of this year's team did precisely that in Problem 4, and thereby reduced their score on that question from 5 out of 7 to 1 out of 7!). In the IMO you have to come up with the ideas, *and implement them accurately*, at the time they are needed. The markers are not interested in ideas which hit you as you leave - or after you have left - the examination room.

We encouraged the six team members (and the reserve in case he was needed) to do all the preparation they could. But with exams and other pressures, the only formal requirement was that they should send in solutions to 5 or so problems every 10-14 days between April 5th and July 5th. On July 6th, shortly before leaving for Moscow, the team met for two days in Birmingham (staying in our house, together with our five children and the Deputy Leader!) to 'warm up' and to generate a sense of purpose and of being part of a team.

The IMO competition consists of two 4 1/2 hour papers taken on successive mornings (15th and 16th July). At least, that was the intention. In the event, the organisation was far from perfect, and the students did not get going on the first morning until after 10.30 a.m. with the result that they were still working at 3 p.m. and did not get their lunch until around 4 o'clock. It also turned out that the invigilation was exceedingly lax, so that the exam rooms were not the easiest places to work in. The students were crammed together in a raked lecture theatre, so that students could see each other's scripts, and if a student in the middle of a row needed extra paper, or had to go to the toilet, s/he had to distyurb everyone else in the row.

Each paper contains just three problems, and each problem is worth 7 marks. The problems are hard, and some are harder than others. I reckon that given this year's papers, most professional mathematicians would have been lucky to have solved two of the problems on Day 1 and one and a bit on Day 2. In addition, the markers this year were very young and exceedingly tough: most were recent winners of either the IMO or of the All Union Olympiad in the old Soviet Union, and they had very clear ideas about what deserved credit (on Problem 5 they were only willing to award 0, 1, 2, or 7 marks!). This meant that many good ideas went unrewarded. A few examples may help to convey just how tough the marking was: one student who had a perfect score (42) last year scored only 24 this year; Robin Michaels scored 29 last year and came around 90th, but this year scored 28 and came 47th; Oliver Johnson scored 17 last year and came 140th, this year he scored 22 and came 86th.

Many teams train together for weeks or months (and in two cases, years) beforehand. Of course, if one is going to enter a competition at the very highest level, then it is certainly important to do enough serious preparation to allow one's students to get the most out of taking part. This is not quite as simple as it sounds. How much is *enough*? And how serious is *serious*? Olympiads are meant to encourage the mathematical development of exceptional students. But there is a clear danger that some countries will become so obsessed with getting their 18 year olds to 'succeed' at the IMO that they forget the true goal of helping these 18 year old students to become effective mathematicians by the age of, say, 25 or 30. They may then adopt training methods which *distort* the mathematical development of their best students.

You might like to have a go at this year's problems and send in your solutions

FIRST DAY

1. Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc-1$.

2. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + (f(x))^2.$$

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

SECOND DAY

4. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

5. Let S be a finite set of points in three dimensional space. Let S_x, S_y, S_z be sets consisting of the orthogonal projections of the points of S onto the yz-plane, the zx-plane, the xy-plane respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$$

where $|A|$ denotes the number of elements in the set A.

6. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive square integers.

(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
(b) Find an integer n such that $S(n) = n^2 - 14$.
(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

Officially the IMO is supposed to be only an individual competition: there is no team competition. But in reality, more attention is paid to the performance of the teams than to this or that individual. The individual performances on the six questions were as follows:

Name	Q1	Q2	Q3	Q4	Q5	Q6	Total	Position (out of 321)	Medal
Oliver Johnson	7	2	0	5	1	7	22	87	Bronze
Robin Michaels	7	7	2	1	7	4	28	47	Silver
Eva Myers	7	4	7	7	2	7	34	14	Gold
Karen Page	7	5	1	5	1	2	21	96	Bronze
Luke Pebody	7	5	7	1	3	5	28	47	Silver
Mark Walters	7	1	6	7	7	7	35	11	Gold

The UK scored 168 out of 252 and so came 5th out of 58 competing countries - behind China (240), the USA (187), Romania (177), and the Commonwealth of Independent States (176), and ahead of Russia (158), Germany (149), Japan (142), Hungary (141), Yugoslavia, France, Poland, Czechoslovakia, Vietnam, Bulgaria, India, Iran, and many others who had in recent years made a habit of beating us. In addition the UK got a surprise prize for submitting the best problem to the IMO - problem number 6.

All in all it was a thoroughly satisfactory performance which was very well received by everyone present.

How the 33rd IMO was run

That the IMO takes place at all is an annual miracle. The host country for a given year declares several years in advance that it wishes to host the IMO for that coming year. The invitation for 1992 was originally issued by the old Soviet Union some 5 years ago, but by the time the event came around, it was Russia who had to pick up the tab. This must have caused many headaches. Russia is in a very painful state of transition at the moment, and we are grateful to them for all their efforts. Invitations were eventually sent out in April/May (six months later than usual). The official invitiation to the UK went astray, and the copy sent to me as Leader arrived at my home address on July 19th, by which time the IMO was over! Fortunately word got round on the grapevine, and a copy was obtained through other channels. Then the DES lost it! But things worked out in the end.

As the IMO expands the cost of running it has increased. The host country

accepts responsibility for all costs of all official participants from landing to take-off. In 1967 just 13 countries took part; in 1979 when the event was held in London, there were just 25 countries. Now, with around 60 countries, the cost of hosting the event is around £1/2 million.

All the practical arrangements are made by the local organisers long in advance. But the body with the formal responsibility for

choosing the problems
marking the scripts and approving the awards
making all other relevant decisions

is the *IMO Jury*, which consists of the Leaders of the participating teams. Each Leader has one vote and noone else may speak or vote at Jury meetings. The 1992 IMO Jury came into existence at its first meeting on 11th July at 9 a.m. and ceased to exist after its last meeting at 11 a.m. on 19th July.

How are the problems chosen? Each participating country is asked to submit up to five problems of an appropriate standard. The resulting collection of 150+ problems is whittled down to a shortlist (this year of around 20 problems) by a committee of experienced Olympiad mathematicians from the host country. The Jury could theoretically reject this shortlist, but in practice they are extremely grateful for this important preparatory work: without it, the delicate task of choosing a set of six problems which is acceptable to all participating countries would be almost impossible. Unfortunately this year the Jury were given not only the shortlisted problems, *but also the solutions*, which resulted in several mistaken judgements as to how hard some of the problems really were. (It is much better if the Jury is left to stew over the problems for a while before being given the solutions. They may then realise that a problem which has a very short, simple solution is actually very hard.)

The Jury works hard. First they eliminate problems

- which are too hard or too easy,
- which have already appeared somewhere in the literature, or
- which have been used in other competitions, or during the training program of some participating country (Leaders are remarkably honest in declaring when this has occurred).

Next, problems which are popular with the majority of members of the Jury are identified and are classified as *Easyish, Middling, and Hardish*. Questions 1 and 4 are usually meant to be slightly easier than the others, while questions 3 and 6 are allowed to be rather hard. This year one of the problems proposed by the UK received the honour of being *Problem number 6*. Little did we then know that we were to receive another surprise honour at the Closing Ceremony.

The whole system depends on trust. There is an awful lot at stake. Leaders have worked long and hard with their teams and want them to do well. And in the day or two before the first paper is sat, they know the problems their students will have to tackle. With modern communications technology it would be easy to cheat if one were determined to do so. Yet the atmosphere is, in general, remarkably open and honest. However, to help avoid temptation the Team Leaders arrive 3 days early and are kept in an entirely separate (and secret) location, away from their teams. This year the Leaders arrived on the 10th July and were put up in Hotel Salyut on the South West edge of Moscow, while the teams arrived on 13th July and were housed in the biggest hotel complex in the world, with over 5 000 beds - Hotel Izmailovo - which was built for the 1980 Olympic Games in the extreme North East of the city. Despite this physical separation, some of the teams met up with their Leaders in the city centre the day before the competition - if only by accident.

When the six problems have been chosen and approved, the Jury has to agree on the precise wording of each question in the four official languages (English, French, Russian and Spanish). Last year the four language groups all produced their own versions and the Jury selected the best version, forcing the other three languages to translate from this 'best' version. That worked very well, but this year the Jury was told first to produce an acceptable version *in*

English, which was then translated into the other languages. This was less satisfactory (especially since it put an extra burden on those who are deemed to be expert in English!), and I hope we will return to the other method next year. Leaders then translate from one of these official languages into the language their students use. It is clearly important that *all* students should get the problems clearly stated in their own language; but *no* student should gain an unfair advantage by being given more information in the version s/he receives. When there are 40 different language versions it is important that they all say the same thing. This can lead to endless wrangling.

Once the 40 odd language versions have been produced, they must all be inspected and checked (e.g. the North Koreans and South Koreans have slightly different versions, but they can read and check each other's).

All this and the other Jury business takes three very full days. On 14 July, while the teams are settling in and the papers are being duplicated for the following day, the Jury was whisked off to visit Sergei Pasad (previously known as Zagorski), an ancient, but very important monastery of the Russian Orthodox church. We returned in the evening and dressed for the Opening Ceremony, which was to take place at the hotel complex where the students were staying. Teams and Leaders are kept separate at the Opening Ceremony, so Leaders can see their teams, but not speak to them in their hour of need - an experience which generates some very strange emotions. We had arranged this year that we would all wear a *UK Team T Shirt* at the Opening Ceremony. Since these were not ready before I had to leave for Moscow, I had to locate the team (without getting too close!) and have them throw mine to me. The Opening Ceremony was a delightful mixture of short speeches, full of Russian idealism (including one from the Russian Minister of Education, and a messaage of greetings from Boris Yeltsin), interspersed with songs and folk dances. The UK Leader made the mistake of sitting in the front row, unaware that the folk dancers had hatched a foul plot to drag such people out to dance in front of everybody - and with my UK Team T Shirt anonymity was denied me. And while I danced, the team was no doubt beginning to feel the butterflies in their tummies at the prospect of the challenge which would face them next morning.

At this stage the teams had not yet really got to know each other. Yet by the time of the Closing Ceremony, less than six days later, when battle had been joined between the contestants and the problems, between the Team Leaders and the Russian Judges who mark the problems, it was as though we were all part of one enormous Mathematical Family.

By the afternoon of the 16th the teams had finished their hard work and could relax and enjoy one another's company. The IMO has two parallel goals: to encourage and challenge those who are especially able in mathematics, and to develop friendship between such students from different countries. So the *apres-ski* is an integral part of the event. The fact that all 350 of them have gone through the same torment of grappling with six hard problems during nine long hours provides a common bond. It also generates nervous tension as students wonder how they fared relative to other teams and to their own hopes and expectations. Our team T shirts served as portable autograph books, collecting signatures and appropriate graffiti from anyone who was invited to sign. (In the middle of the central design on the front of Karen's T shirt is a signature with the comment: "I really enjoyed writing this!" - all the more amusing since I was assured that she wasn't wearing the T shirt at the time.)

The scripts are first marked by the Leader and Deputy Leader. This can be a tough job: six students working for nine hours each can generate an awful lot of mathematics, not all of which is easy to understand! There is then a hectic schedule of official Judging, which allocates half an hour per question per team (though sometimes this can stretch to 3 hours or more!).

The Judging this year was done by a team of 45 enthusiastic young Russians under the excellent leadership of Arkady Slinko. They were superb, though

very strict. Time and again one would see this or that judge, who should rightly have been taking a break, join in the attempt to make sense of some student's script. They positively *wanted* to be involved in anything mathematical; and if the script contained an interesting solution, so much the better. In such a setting it is important to be thoroughly well prepared, so that one can explain to someone whose mother tongue is not English what the script which is being judged actually contains. This can be especially difficult because if the script is the right way up for the judges to see it, then it is upside down for the person who is trying to explain what it says! In particular, one does not score marks at an IMO for making a *brave attempt*. The IMO is about *solving* problems, not failing to solve them. A failed soution may contain some lovely ideas, but if they don't help to solve the problem, they are unlikely to score more than one, or occasionally two marks.

Our team goals were to raise our minimum individual score from last year's minimum of 17 to ≥ 20. Before the IMO I had also told them that there was no reason why we should not return with 4 Silver and 2 Bronze medals. In the event, though the marking was far tougher than we had expected, the team did even better, with 2 Golds (Mark and Eva), 2 Silvers (Robin and Luke), and 2 Bronzes (Oliver and Karen). We had no Michael Fryers to achieve an *individual* perfect score of six sevens; but the team managed six sevens between them on Question 1 - one of only two teams to achieve this (the other was Yugoslavia). And not only were all our individual scores of ≥ 21, but the team as a whole scored ≥ 23 on each individual question. This consistency was the key to our success, with the team coming 5th overall - one of the best results for years.

When the team realised that we were in with a chance of something unusual, they set out to corner the market for the Russian equivalent of champagne. As it sank (soaked?) in just how remarkably they had all performed, and how close we had come to being second or third, one could see the tension in their faces begin to evaporate. But when they started to sing God Save the Queen in the dining room, I too evaporated.

We suppress most of the details. Indoor French cricket was quite a hit(!), but night time cricket in the car park outside the hotel merely led to the hasty loss of our supply of balls. New Zealand again proved that they are a social, if not exactly a mathematical threat (the Ireland team beat them by 1 point); it was not for nothing that they were christened (after the initials NZL) *"the Nuzzles"* - being more interested in 'triangles' a la *Barbara Cartland* than in plane Euclidean ones. And when the soldiers guarding the Kremlin refused to let us take our celebration cake in with us, we all set to and consumed it under their noses before taking it in one piece at a time!

At the Closing Ceremony, the UK Leader received a surprise award for having submitted the "best" problem at this 33rd IMO. But this time it was the Swedish Leader and the Norwegian Deputy who were hauled up on stage to dance. Russia is a country with curious ways of doing (and of *not* doing) things. It's future is balanced on a knife edge. But it is a country of surprises.

We may have been gone only ten days, but I swear that during those ten days I was privileged to see six lovely lumps of human potential mature into - well, time will tell what. I hope they will all look back on the experience as having contributed positively to their mathematical and social development.

And having roasted us during the judging process, the Russian judges set about "toasting us to death". The Closing Banquet consisted of one long series of toasts - many of them to future Anglo-Russian friendship. I did my best to stay with them, but had clearly not trained sufficiently seriously, and went to bed at 4 a.m.. By that time the Canadian and USA teams were midway through a challenge match of *Ultimate* (team frisbee) in the car park, prior to leaving for their flight at 5.30 a.m..

Oh by the way - next year's IMO will take place in Istanbul.